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A Comparative Study of AG*-groupoids

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ABSTRACT

A magma satisfying the identity $(ab)c=(cb)a$ is called an LA-semigroup or AG-groupoid. We investigate the concept of AG-groupoid, as a subclass of AG-groupoids and establish some relations of AG*-groupoids with other known subclasses of AG-groupoid, like: Stein, Stein*, Jordan and Schroder AG-groupoids. Furthermore, we use the latest computational techniques of Mace-4 and Prover-9 to provide various examples and counterexamples for our relevant investigation.*

1. Introduction

Like the associative and commutative structures, the non-associative structures also arise in various situations. The Cayley octonions [1], non-associative loop arising in "coordinatization of projective planes" and in the "Einstein velocity addition in relativity theory" are some well-known examples of non-associative structures. The field of non-associative structures includes groupoids, quasigroup and loops, non-associative semirings, LA-semigroups or AG-groupoids and self-distributive algebras [2]. The concept of non-associative algebra emerged in 1930's during achievements of Moufang, Mordorch, Bol, Sushkevich, and many more [3] and one of the earliest studies [4]. Some comprehensive materials on the Jordan algebras, containing significant literature on "general non-associative algebras" are [5-7], which made the field more interesting and attractive.

Kazim and Naseer [8] introduced a new structure as a generalization of commutative semigroup in 1972 and called it a "left almost semigroup (or LA-semigroup)". LA-semigroup (or AG-groupoid) is in general a non-associative groupoid satisfying the "left inventive law: $(ab)c=(cb)a$."Every AG-groupoid G satisfies the "medial law [9], $(ab)(cd)=(ac)(bd)$". It is also proved in [9] that " if G contains a left identity, then it is unique" and such an AG-groupoid is known an AG-monoid. "An AG-groupoid G is called AG-band, if all its elements are idempotent" [10]. An AG-groupoid G, in which the identity"$(aa)a=a(aa)=a$ holds, is called AG-3-band" [11]. G is called AG* if it satisfies the property, $(ab)c=b(ac)\ \forall a,b,c\in G$ .Throughout the article, we shall denote an AG-groupoid simply by G otherwise stated else. We shall use uv for $u\cdot v$, $uv\cdot wt$ for $(uv)(wt)$ and $(uv\cdot w)t$ for $((uv)w)t$ to avoid excessive use of parenthesization and dots.

AG-groupoids have various applications in flock theory, matrix algebra, finite mathematics, fuzzy mathematics and topology, soft and cubic sets [12-19]. Various ideals are defined for a variety of subclasses of AG-groupoids. Fuzzification of the field has also made the field very useful and interesting. In the following table, we list some of the previously known classes of AG-groupoids [20-23] with their identities that are required for our study in the rest of this article.

2. Material and Method

We use the modern computational techniques of GAP, Prover-9 and Mace-4 to investigate AG*-groupoid. We use these tools to produce various examples and counterexamples for the investigation of our subclass of AG*-groupoid. In the following, were call the definition of an AG*-groupoid and give some of its known relations with some subclasses of AG-groupoids that are required for further investigation of our interest.

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Table 1. Various AG-groupoids with their identities

AG-groupoid	Satisfying identity	AG-groupoid	Satisfying identity
Stein	$a(bc)=(bc)a$	right nuclear square	$(ab)c^2=a(bc^2)$
paramedial	$(ab)(cd)=(db)(ca)$	right [left] alternative	$b(aa)=(ba)a\ [(aa)b=a(ab)]$
Jordan	$a(b^2c)=b^2(ac)$	LC[RC]	$(ab)c=(ba)c\ [a(bc)=a(cb)]$
Stein*	$a(ab)=ba$	Schroder	$a(ab)=(ab)a$
left nuclear square	$a^2(bc)=(a^2b)c$	Completely inverse	$(ab)a=a,(ba)b=b,\ ab=ba$
middle nuclear	$(ab^2)c=a(b^2c)$	Rectangular*	$(ab)(cd)=(ad)(cb)$
Bol*	$(ab\cdot c)d=a(bc\cdot d)$	unipotent	$a^2=b^2$
flexible	$(ab)a=a(ba)$	slim	$a(bc)=ac$
LP	$b(ac)=a(bc)$	RP	$(ab)c=(ac)b$

Definition 1: "A groupoid G is called an AG-groupoid, if it satisfies the identity.

$$(ab)c=(cb)a\ \forall a,\ b,\ c\in G\text{.”} \qquad (2.1)$$

Definition 2: Every AG-groupoid satisfies the medial law

$$(ab)(cd)=(ac)(bd)\ \forall a,\ b,\ c,d\in G\text{.”} \qquad (2.2)$$

Definition 3: "An AG-groupoid G is called an AG*-groupoid, if it satisfies the identity.

$$(ab)c=b(ac)\ \forall a,\ b,\ c\in G\text{.”} \qquad (2.3)$$

Theorem 1 [24]: "Every AG*-groupoid is Bol*-AG-groupoid".

Proposition 1: [24]: "Every Bol* - AG-groupoid is para-medial AG-groupoid".

Theorem 2 [24]: "Every paramedial AG-groupoid is left nuclear square AG-groupoid".

Corollary 1: **"**Every AG*-groupoid is paramedial AG-groupoid".

Theorem 3 [24]: "Let G be an AG*-groupoid. Then, G is nuclear square AG-groupoid."

Theorem 4 [24]: "An AG*-groupoid having a left cancellative element is T^1-AG-groupoid".

Theorem 5 [24]: "Every AG*-groupoid is left alternative AG-groupoid".

Theorem 6 [25]: **"**Every AG*-groupoid is rectangular*-AG-groupoid".

Theorem 7 [26]: **"**Every AG*-groupoid is right commutative-AG-groupoid".

For counter examples of the above cited results, the reader is referred to [16, 23-26].

3. Results and Discussion

Here, we precisely investigate AG*-groupoid and give some of its relations with other known subclasses. Throughout this note, G will represent an AG*-groupoid otherwise stated else. We start with the following theorem:

Theorem 8: Every AG*-groupoid G is Jordan.

Proof: Let $a,b,c\in G.$ Then by (2.2-2.3), we have

$$a(bb\cdot c)=(bb\cdot a)c=(b\cdot ba)c=ba\cdot bc=bb\cdot ac.$$

Thus, $a\left(b^2\cdot c\right)=b^2\left(ac\right)$. Hence, G is Jordan.

Next, we provide a counterexample to depict that converse of the previous theorem is not valid.

Example 1: Let $S=\{0,1,2\}$, then $(S,*)$ as given in the following Example 1 is an example of Jordan AG-groupoid that is not an AG*-groupoid.

*	0	1	2
0	1	1	2
1	2	2	2
2	2	2	2

Example 1

Clearly, $(0*0)*1\neq 0*(0*1)$

Theorem 9: For any AG*-groupoid G, the following properties are satisfied.

i. $(a\cdot bc)d=ab\cdot cd$

ii. $((ab\cdot c)d)e=(a(bc\cdot d))e$

iii. $(a(bc\cdot d))e=ab(cd\cdot e)$

iv. $((ab\cdot c)d)e=ed(a\cdot bc)$

Proof: For an AG*-groupoid G, we prove the properties one by one, as follows:

i. $(a \cdot bc)d = ab \cdot cd$

Let a, b, c, d in G. Then by (2.1-2.3) and Corollary 1, we have

$$\begin{aligned}(a \cdot bc)d &= (ba \cdot c)d = dc \cdot ba \\ &= ac \cdot bd = ab \cdot cd.\end{aligned}$$

Thus $(a \cdot bc)d = ab \cdot cd$.

ii. Let a, b, c, d, e be elements of G. Then, again by (2.1-2.3), we get

$$\begin{aligned}((ab \cdot c)d)e &= (dc \cdot ab)e = (da \cdot cb)e \\ &= ((c \cdot da)b)e = ((b \cdot da)c)e \\ &= (da \cdot bc)e = ((bc \cdot a)d)e = (a(bc \cdot d))e.\end{aligned}$$

Thus $((ab \cdot c)d)e = (a(bc \cdot d))e$.

iii. $(a(bc \cdot d))e = ab(cd \cdot e)$

Again, let $a, b, c, d, e \in G$. Then by (2.1-2.3), we have

$$\begin{aligned}(a(bc \cdot d))e &= ((bc \cdot a)d)e = ((c \cdot ba)d)e \\ &= ((d \cdot ba)c)e = ((bd \cdot a)c)e \\ &= (ca \cdot db)e = (cd \cdot ab)e = ab(cd \cdot e).\end{aligned}$$

Equivalently $(a(bc \cdot d))e = ab(cd \cdot e)$.

iv. To prove $((ab \cdot c)d)e = ed(a \cdot bc)$

Let $a, b, c, d, e \in G$. Then, by (2.1-2.2), we have

$$\begin{aligned}((ab \cdot c)d)e &= ed \cdot (ab \cdot c) = ed(cb \cdot a) \\ &= ed(b \cdot ca) = (b \cdot ed) \cdot ca \\ &= (ca \cdot ed)b = ed(ca \cdot b) \\ &= ed(ba \cdot c) = ed(a \cdot bc).\end{aligned}$$

Thus $((ab \cdot c)d)e = ed(a \cdot bc)$.

However, the converse of each of the above listed properties is not true. We provide a counterexample for these properties as follows.

Example 2: Let $S = \{0,1,2\}$, then $(S,*)$ in the following Example 2, satisfies the above listed properties from (i-iv) but, is not an AG*-groupoid.

*	0	1	2
0	1	1	1
1	2	2	2
2	2	2	2

Example 2

Next, we provide an example that is neither an LC-AG-groupoid nor an AG*-groupoid is a Stein AG-groupoid. However, combination of these two properties gives Stein AG-groupoid as given in the following result.

Example 3: Let $Q = \{1,2,3,4,5,6\}$ then $(Q,*)$ in Example 3(i) is LC-AG-groupoid but not a Stein and $(Q,\cdot)$ in Example 3(ii) is AG*-groupoid that is not a Stein-AG.

$*$	1	2	3	4	5	6
1	2	2	2	2	2	2
2	3	3	3	3	3	3
3	3	3	3	3	3	3
4	2	2	2	2	2	2
5	2	2	2	2	2	2
6	2	2	2	2	2	2

Example 3 (i)

$\cdot$	1	2	3	4	5	6
1	3	4	5	5	5	5
2	3	4	6	6	5	5
3	5	5	5	5	5	5
4	6	6	5	5	5	5
5	5	5	5	5	5	5
6	5	5	5	5	5	5

Example 3 (ii)

Theorem 10: Every LC-AG*-groupoid is a Stein.

Proof: Let G be an LC-AG*-groupoid and $a, b, c \in G$. Then by (2.1, 2.3) and by the property of LC-AG-groupoid

$$a(bc) = (ba)c = (ab)c = (cb)a = (bc)a.$$

Thus $a(bc) = (bc)a$. Hence G is Stein.

The converse implication of Theorem 10 may not be true, as shown in the following counterexample.

Example 4: Let $S = \{0,1,2,3,4\}$, then $(S,\cdot)$ with the following Example 4 is Stein AG-groupoid but not an *AG**.

$\cdot$	0	1	2	3	4
0	2	2	3	4	4
1	3	3	4	4	4
2	3	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4

Example 4

Theorem 11: Every idempotent-AG*-groupoid is Schroder.

Proof: Let G be an idempotent-AG* and $p, q \in G.$ Then by Eqn. 2.2 and 2.3 and idempotent property, we have

$$p(pq) = (pp)q = (pp)(qq) = (pq)(pq) =$$
$$= q(p \cdot pq) = q(pp \cdot q) = q(pq) = (pq)q.$$

Thus, $p(pq) = (pq)q.$ Hence the result is proved.

However, converse of Theorem 11 may not be true as shown below.

Example 5: Let $T = \{0, 1, 2\}$, then $(T, \cdot)$ as given in the Example 5 below is Schroder, but not AG*, as $(1 \cdot 1) \cdot 0 \neq 1 \cdot (1 \cdot 0)$

·	0	1	2
0	1	2	2
1	2	0	2
2	2	2	2

Example 5

Theorem 12: Every AG*-groupoid having an idempotent element is a Stein*.

Proof: Let $p, q \in G,$ such that p is idempotent, then by Eqn. 2.1 and 2.2

$$p \cdot pq = p(pp \cdot q) = p(qp \cdot p) = (qp \cdot p)p$$
$$= (pp \cdot q)p = (pq)p = q(pp) = qp$$

Thus the result follows.

Example 6: Let B= {0, 1, 3, 3} $B = \{0, 1, 2, 3\}$, then *(B,)* is Stein*-AG-groupoid that is not AG* as, $(0 \cdot 0) \cdot 1 \neq 0 \cdot (0 \cdot 1).$

·	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

Example 6

A slim AG-groupoid may or may not be an AG*-groupoid. However, the result holds if each of its elements is idempotent.

Theorem 13: Every idempotent slim-AG-groupoid is AG*.

Proof: Let G be a right slim-AG-groupoid having an idempotent element and let $a, b, c \in G.$ Then

$$\begin{aligned} b \cdot ac &= bc && \text{(using slim AG-groupoid)} \\ &= bb \cdot c && \text{(using idempotent law)} \\ &= cb \cdot b && \text{(using left invertive law)} \\ &= (c \cdot ab)b && \text{(using slim AG-groupoid)} \\ &= (c \cdot ab)(ab) && \text{(using slim AG-groupoid)} \\ &= (ab \cdot ab)c && \text{(using left invertive law)} \\ &= ab \cdot c && \text{(using idempotent law)} \end{aligned}$$

Thus $b \cdot ac = ab \cdot c.$ Hence every silm AG-groupoid with an idempotent element is AG*-groupoid.

In Example 3 Table (ii) shows that unipotent AG-groupoid is not flexible. We also provide a counterexample to show that AG*-groupoid is not flexible. However, the combination of these two classes gives a flexible AG-groupoid.

Example 7: For $G = \{u, v, w\}, (G, \cdot)$ is an AG*-groupoid that is not flexible.

·	u	v	w
u	v	w	u
v	u	v	w
w	w	u	v

Example 7

Theorem 14: Every AG*-groupoid having unipotent property is flexible.

Proof: Let G be an unipotent AG* and $g, h \in G,$ then by (2.1, 2.3), we have

$$gh \cdot g = h \cdot gg = h \cdot hh = hh \cdot h = gg \cdot h = hg \cdot g = g \cdot hg.$$

Thus $gh \cdot g = g \cdot hg$ and hence G is flexible.

Theorem 15: An AG*-groupoid G is a semigroup if G also satisfies any of the following properties.

i. G is an LP
ii. G is an RP
iii. G is self-dual
iv. G is Stein.

Proof: Let G be an AG*-groupoid and $p, q, r \in G.$ Then

i. By property of LP-AG-groupoid and AG*-groupoid, we have
$$pq \cdot r = q \cdot pr = p \cdot qr \Rightarrow pq \cdot r = p \cdot qr.$$

ii .By property of RP-AG-groupoid and AG*-groupoid, we have
$$pq \cdot r = pr \cdot q = qr \cdot p = qp \cdot r = p \cdot qr.$$

iii. By property of self-dual and AG*-groupoid, we have

$$p\cdot qr = r\cdot qp = qr\cdot p = pr\cdot q$$
$$= r\cdot pq = q\cdot pr = pq\cdot r$$

iv. By property of Stein and AG*-groupoid, we have

$$p\cdot qr = qr\cdot p = pr\cdot q = r\cdot pq = pq\cdot r.$$

Thus in each case $p\cdot qr = pq\cdot r.$ Hence the result proved.

3.1. Constructions Involving AG*-groupoids

Constructions for algebraic structures play a key role in their development. The examples and counterexamples constructed by this method are usually not even easily possible through computer. Several open and hard problems are usually solved by the method of constructing examples. A variety of constructions is provided for achieving quasigroups and loops etc. It has also been noticed that the methodology of construction can be implemented in computers in order to make the job easier. For instance, a simple way of construction is used in GAP for constructing AG-groups from the known abelian groups in [27]. We discuss some constructions for our class of AG* groupoids.

Theorem 16: Let $(G,\cdot)$ be an AG*-groupoid and $r,s,t\in G$. Define $*$ on G as $r*s=(rc)s,$ where c is any fixed element in G, then $(G,*)$ is a commutative semigroup.

Proof: Obviously,

$$r*s=(rc)s$$
$$=(sc)r = s*r.$$

Thus $(G,*)$ is commutative. Next

$$
\begin{aligned}
(r*s)*t &= ((rc\cdot s)c)t \\
&= ((sc\cdot r)c)t && \text{(By left invertive law)} \\
&= ((c\cdot sr)c)t && \text{(By AG*-groupoid)} \\
&= (sr\cdot cc)t && \text{(By AG*-groupoid)} \\
&= (sc\cdot rc)t && \text{(By medial law)} \\
&= (rc)(sc\cdot t) && \text{(By AG*-groupoid)} \\
&= r*(s*t)
\end{aligned}
$$

Thus $(G,*)$ is a commutative semigroup.

Theorem 17: Let $(G,\cdot)$ be a paramedial AG-groupoid, define $*$ on G as $p*q=p(cq),$ for some fixed $c\in G$ then, $(G,*)$ is AG-groupoid. Further, if $(G,\cdot)$ is an AG*-groupoid then $(G,*)$ is also an AG*.

Proof: Let $p,q,r\in G$ then by definition of $*$ we have

$$(p*q)*r = (p(cq))(cr) = (r(cq))(cp) = (r*q)*p.$$

Thus $(G,*)$ is an AG-groupoid.

Now let $(G,\cdot)$ be an AG*-groupoid. Then

$$(p*q)*r = (p(cq))(cr) = (cq)(p(cr))$$
$$= q(c(p(cr))) = q*(p*r).$$

Thus $(G,*)$ is an AG*-groupoid.

The following Venn diagram depicts the relation of various subclasses of AG-groupoids with AG*-groupoid.

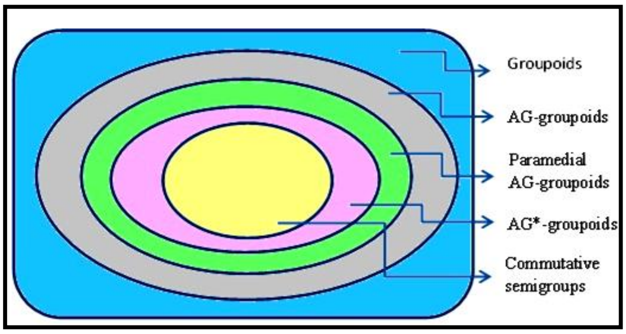

Fig. 1. Relation of various subclasses of AG-groupoids with AG*

4. Conclusion

Many new subclasses of AG-groupoids are introduced and investigated recently. In this article, we have further investigated AG*-groupoid and have established various relations with some known subclasses of AG-groupoid. We proved that, any AG*-groupoid is Jordan, and that when an idempotent element is added to AG*-groupoid, it becomes a Schroder, and a Stein*. Moreover, it is investigated that every LC-AG*-groupoid is a stein, and every AG*-groupoid is a right slim. Finally, we proved that every unipotent AG*-groupoid is flexible. Furthermore, we produced several examples and counterexamples using the latest computational techniques of Mace-4 and GAP to improve the quality of this article.

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