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ANALYTIC SOLUTION FOR PIPE FLOW OF AN OLDROYD 8-CONSTANT FLUID

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This paper investigates the pipe flow of an Oldroyd 8-constant fluid. The governing nonlinear equations are first modelled and then solved analytically by utilizing homotopy analysis method (HAM). The convergence of the developed series solution is established. The influence of the important parameters of interest is seen on velocity.

Keywords: Oldroyd 8-constant fluid, HAM solution, Nonlinear problem

1. Introduction

It is a known fact that complex nature of non-Newtonian fluids cannot be described by a single constitutive equations. Because of this reason, several constitutive equations have been suggested for the non-Newtonian fluids. Amongst these fluid models, differential type fluids have attracted much attention as well as controversies. Another type of fluids are rate type fluids. The very simple subclasses of rate type fluids are the Maxwell and Oldroyd 3-constant fluids. But these fluids do not include rheological effects when steady unidirectional flows are considered. However, an Oldroyd 8-constant fluid includes such effects even for unidirectional steady flow. Keeping various aspects of non-Newtonian fluids, several investigators [1-10] are engaged in studying the flows as the initial and boundary value problems.

The purpose of the present communication is two fold. Firstly to consider the pipe flow of an Oldroyd 8-constant fluid. Secondly to obtain the analytic solution of the non-linear differential system using homotopy analysis method [11]. The HAM is a powerful mathematical technique for solving nonlinear problems and have already been applied for the solution of many nonlinear problems [12-22]. This paper is organized as follows. In section 2 the problem is formulated. The analytic solution is presented in section 3. The convergence of the obtained solution is developed in section 4. The results of pertinent parameters on the flow are also included in the same section. Section 5 comprises the concluding remarks.

2. Mathematical Analysis

Let us consider the steady flow of an Oldroyd 8-constant fluid in a circular pipe. The z-axis is taken along the axis of the flow. The flow in the pipe is induced due to constant applied pressure gradient in the z-direction. Defining

$$\mathbf{V} = [0, 0, u(r)] \tag{1}$$

the incompressibility condition is automatically satisfied and *r* and *z* components of momentum equation yield

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{d}{dr}\left(rS_{rr}\right) - \frac{S_{\theta\theta}}{r}, \tag{2}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{d}{dr}\left(rS_{rz}\right), \tag{3}$$

where for an Oldroyd 8-constant fluid the constitutive equation for the Cauchy stress is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}. \tag{4}$$

In the above equation *p* is the pressure, **I** is the identity tensor and the extra stress tensor **S** is given by

$$\begin{aligned}&\mathbf{S} + \lambda_1 \frac{D\mathbf{S}}{Dt} + \frac{\lambda_3}{2}\left(\mathbf{S}\mathbf{A}_1 + \mathbf{A}_1\mathbf{S}\right)\\ &+ \frac{\lambda_5}{2}\left(\mathrm{tr}\mathbf{S}\right)\mathbf{A}_1 + \frac{\lambda_6}{2}\left[\mathrm{tr}(\mathbf{S}\mathbf{A}_1)\right]\mathbf{I}\\ &= \mu\left[\mathbf{A}_1 + \lambda_2\frac{D\mathbf{A}_1}{Dt} + \lambda_4\mathbf{A}_1^2 + \frac{\lambda_7}{2}\left[\mathrm{tr}\left(\mathbf{A}_1^2\right)\right]\mathbf{I}\right],\end{aligned} \tag{5}$$

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in which

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}}, \quad \mathbf{L} = \mathrm{grad}\mathbf{V} \tag{6}$$

and $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, λ_i (i=1 to 7) are material parameters of the fluid and are assumed constants. The contravariant convected derivative D/Dt for steady flow is defined as

$$\frac{D\mathbf{S}}{Dt} = (\mathbf{V}\cdot\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}. \tag{7}$$

The stress is considered as

$$\mathbf{S}(r) = \begin{pmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{pmatrix}. \tag{8}$$

Making use of Eqs. (1) and (8) in Eq. (5) one obtains

$$S_{rr} + (\lambda_3 + \lambda_6) S_{rz} \frac{du}{dr} = \mu(\lambda_4 + \lambda_7)\left(\frac{du}{dr}\right)^2, \tag{9}$$

$$S_{r\theta} + \frac{\lambda_3}{2} S_{z\theta} \frac{du}{dr} = 0, \tag{10}$$

$$S_{rz} + \frac{(\lambda_3 + \lambda_5)}{2}(S_{rr} + S_{zz})\frac{du}{dr} + \frac{\lambda_5}{2} S_{\theta\theta}\frac{du}{dr} - \lambda_1 S_{rr}\frac{du}{dr} = \mu\frac{du}{dr}, \tag{11}$$

$$S_{\theta\theta} + \lambda_6 S_{rz}\frac{du}{dr} = \mu\lambda_7\left(\frac{du}{dr}\right)^2, \tag{12}$$

$$S_{\theta z} + \frac{(\lambda_3 - 2\lambda_1)}{2} S_{\theta r}\frac{du}{dr} = 0, \tag{13}$$

$$S_{zz} + (\lambda_3 + \lambda_6 - 2\lambda_1) S_{rz}\frac{du}{dr} = \mu(\lambda_4 + \lambda_7 - 2\lambda_2)\left(\frac{du}{dr}\right)^2, \tag{14}$$

$$S_{r\theta} = S_{\theta z} = 0, \tag{15}$$

$$S_{rr} = \frac{\left[\begin{array}{l}\mu(\lambda_4 + \lambda_7 - \lambda_3 - \lambda_6)\left(\frac{du}{dr}\right)^2 \\ + \mu\{(\lambda_4 + \lambda_7)\alpha_2 - (\lambda_3 + \lambda_6)\alpha_1\}\left(\frac{du}{dr}\right)^4\end{array}\right]}{1 + \alpha_2\left(\frac{du}{dr}\right)^2}, \tag{16}$$

$$S_{rz} = \frac{\mu\frac{du}{dr} + \mu\alpha_1\left(\frac{du}{dr}\right)^3}{1 + \alpha_2\left(\frac{du}{dr}\right)^2}, \tag{17}$$

$$S_{\theta\theta} = \frac{\mu(\lambda_7 - \lambda_6)\left(\frac{du}{dr}\right)^2 + \mu(\lambda_7\alpha_2 - \lambda_6\alpha_1)\left(\frac{du}{dr}\right)^4}{1 + \alpha_2\left(\frac{du}{dr}\right)^2}, \tag{18}$$

$$\alpha_1 = \lambda_1(\lambda_4 + \lambda_7) - (\lambda_3 + \lambda_5)(\lambda_4 + \lambda_7 - \lambda_2) - \frac{\lambda_5\lambda_7}{2},$$
$$\alpha_2 = \lambda_1(\lambda_3 + \lambda_6) - (\lambda_3 + \lambda_5)(\lambda_3 + \lambda_6 - \lambda_1) - \frac{\lambda_5\lambda_6}{2}.$$

Now first the velocity field is determined from Eq. (3) and then the pressure field can be easily calculated using Eq. (2). The relevant boundary conditions are

$$\left|\frac{du}{dr}\right| < \infty \quad \text{at} \quad r = 0, \tag{19}$$

$$u = 0 \quad \text{at} \quad r = R. \tag{20}$$

$$\bar{r} = \frac{r}{R}, \quad \bar{u} = \frac{\mu}{\left(\frac{\partial p}{\partial z}\right)R^2}u, \quad \bar{\alpha}_1 = \frac{\alpha_1\left(\frac{\partial p}{\partial z}\right)^2}{\mu^2}, \quad \bar{\alpha}_2 = \frac{\alpha_2\left(\frac{\partial p}{\partial z}\right)^2}{\mu^2}, \quad \bar{S}_{rz} = \frac{S_{rz}}{\left(\frac{\partial p}{\partial z}\right)R}, \tag{21}$$

Equations (3), (8) and (9) become

$$\frac{d}{dr}(rS_{rz}) - r = 0, \tag{22}$$

$$|u'(0)| < \infty, \tag{23}$$

$$u(1) = 0, \tag{24}$$

$$S_{rz} = \frac{u' + \alpha_1(u')^3}{1 + \alpha_2(u')^2}, \tag{25}$$

where bars have been suppressed for simplicity. Integrating Eq. (22) one obtains

$$r\left[\frac{u'+\alpha_1(u')^3}{1+\alpha_2(u')^2}\right]-\frac{r^2}{2}=C, \tag{26}$$

where C is a constant of integration and can be calculated using condition (23) and is given by

$$C=0. \tag{27}$$

Equation (26) takes the form

$$2\alpha_1(u')^3-\alpha_2 r(u')^2+2u'-r=0, \tag{28}$$

Equation (28) subject to boundary condition (24) has been solved using homotopy analysis method in the next section.

3. HAM Solution

3.1 Zeroth-order deformation equation

The function $u(r)$ can be expressed by the set of base functions

$$\left\{r^k \,\middle|\, k\geq 0\right\} \tag{29}$$

in the form

$$u(r)=\sum_{k=0}^{\infty} a_{m,k} r^k, \tag{30}$$

where $a_{m,k}$ are the coefficients. By considering the *Rule of solution expressions* for $u(r)$ and Eqs. (24) and (28) one can choose

$$u_0(r)=\frac{1}{4}\left(r^2-1\right) \tag{31}$$

as the initial approximation of $u(r)$

$$\mathbf{L}(u)=u' \tag{32}$$

is the auxiliary linear operator and

$$\mathbf{L}[C_1]=0, \tag{33}$$

where C_1 is an arbitrary constant.

Eq. (28) suggests that the nonlinear operator is of the form

$$\mathbf{N}[\bar{u}(r,p)]=2\frac{\partial\bar{u}(r,p)}{\partial r}-r-\alpha_2\left(\frac{\partial\bar{u}(r,p)}{\partial r}\right)^2+\alpha_1\left(\frac{\partial\bar{u}(r,p)}{\partial r}\right)^3. \tag{34}$$

The zeroth order deformation problem can be constructed by taking a non-zero auxiliary parameter $\hbar$

$$(1-p)\mathbf{L}\left[\bar{u}(r,p)-u_0(r)\right]=p\hbar\mathbf{N}\left[\bar{u}(r,p)\right], \tag{35}$$

$$\bar{u}(1,p)=0, \tag{36}$$

where $p\in[0,1]$ is the embedding parameter. For p = 0 and p = 1, one respectively has

$$\bar{u}(r,0)=u_0(r),\quad \bar{u}(r,1)=u(r). \tag{37}$$

When p increases from 0 to 1, $u(r,p)$ varies continuously from initial guess $u_0(r)$ to the final solution $u(r)$. By Taylor's theorem and Eq. (37) one can write

$$\bar{u}(r,p)=u_0(r)+\sum_{m=1}^{\infty}u_m(r)\,p^m, \tag{38}$$

$$u_m(r)=\frac{1}{m!}\left.\frac{\partial^m\bar{u}(r,p)}{\partial p^m}\right|_{p=0} \tag{39}$$

and convergence of series (38) depends upon $\hbar$. Assume that $\hbar$ is selected such that the series (38) is convergent at p = 1, then due to Eq. (37) one get

$$u(r)=u_0(r)+\sum_{m=1}^{\infty}u_m(r). \tag{40}$$

3.2. mth-order deformation equation

Differentiating m times the zeroth order deformation Eq. (35) with respect to p and then dividing by $m!$ and finally setting p = 0 the following mth-order deformation problem can be obtained

$$\mathbf{L}\left[u_m(r)-\chi_m u_{m-1}(r)\right]=\hbar\mathbf{R}_m(r), \tag{41}$$

$$u_m(1)=0, \tag{42}$$

$$\mathbf{R}_m(r) = 2u'_{m-1}(r) - (1-\chi_m)r$$
$$-\alpha_2 r \sum_{k=0}^{m-1} u'_{m-1-k}(r)u'_k(r) \tag{43}$$
$$+2\alpha_1 \sum_{k=0}^{m-1} u'_{m-1-k}(r) \sum_{l=0}^{k} u'_{k-l}(r)u'_l(r),$$

in which

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{44}$$

Let $u_m^*(r)$ denotes a special solution of Eqs. (41) and (42) and using Eq. (33) one can write the general solution

$$u_m(r) = u_m^*(r) + C_1, \tag{45}$$

and C_i can be determined by using the boundary condition (31) and is given by

$$C_1 = -u_m^*(r)\Big|_{r=1}.$$

This is an easy way to solve linear Eqs. (41) subject to conditions (42) in the order m = 1,2,3,... with the help of symbolic computation software MATHEMATICA.

4. Analysis of the Results

The convergence of the series (40) strongly depends upon the value of the auxiliary parameter $\hbar$ as mentioned by Liao [11]. The valid region of the values of the parameter $\hbar$ can be obtained by plotting the $\hbar$-curve. The $\hbar$-curves for different values of the fluid parameters α_1 and α_2 for 15th order of approximation are displayed in Figs. 1 and 2. In Fig. 1 the influence of parameter α_1 on the value of $\hbar$ is shown when α_2 is kept fixed. It is evident from Fig. 1 that the range of $\hbar$ is $-1<\hbar<0$ and with an increase in parameter α_1 the admissible range shrinks towards zero. Fig. 2 gives the variation in the admissible range of $\hbar$ when one vary α_2 keeping α_1 fixed. Fig. 2 elucidates that the range of $\hbar$ is $-1.3<\hbar<0$ and the admissible range interval is stretched by an increase in α_2. To see the effects of parameters α_1 and α_2 on the velocity Figs. 3 and 4 are plotted. Figure 3 depicts that the velocity and boundary layer thickness increases by increasing the parameter α_1. However the effects of α_2 are quite opposite to that of α_1.and are shown in Fig. 4.

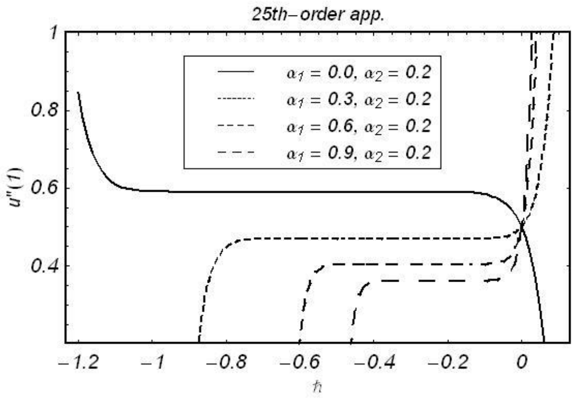

Figure 1. $\hbar$-curves for different values of fluid parameter α_1.

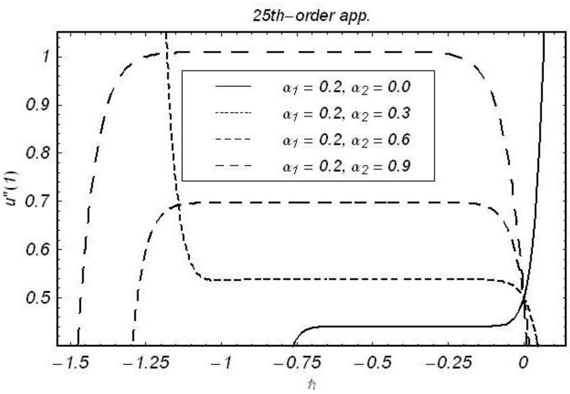

Figure. 2. $\hbar$-curves for different values of fluid parameter α_2.

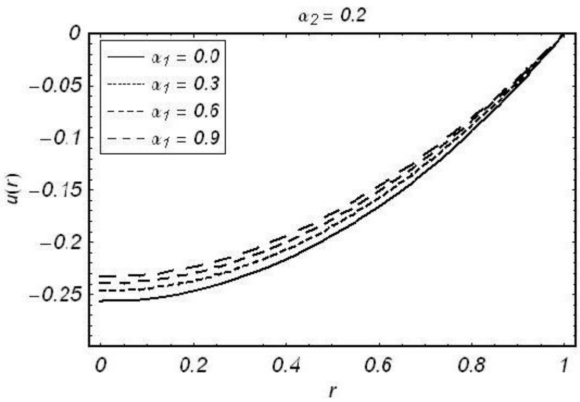

Figure 3. Influence of parameter α_1 on the velocity.

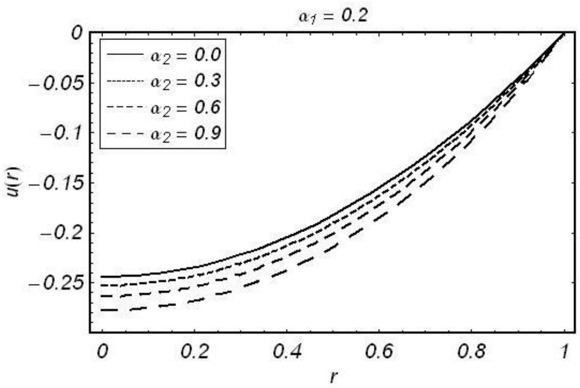

Fig. 4. Influence of parameter α_2 on the velocity.

5. Conclusions

This paper deals with the analytic solution for the pipe flow of an Oldroyd 8-constant fluid. The analytic solution has been obtained for the governing problem using homotopy analysis method. The convergence of the series solution is explicitly discussed. The results are sketched and discussed for the variations of the fluid parameters. It is found that both the fluid parameters α_1 and α_2 have opposite effects on the velocity profile. The corresponding results of Oldroyd 3-constant ($\lambda_3=\lambda_4=\lambda_5=\lambda_6=\lambda_7=0$) and Oldroyd 6-constant fluids ($\lambda_6=\lambda_7=0$) can be obtained from the presented solution as special cases.

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