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A Comparative Study of Anti-rectangular AG-groupoids and Commutative Semigroups

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ABSTRACT

In this paper, anti-rectangular AG-groupoids are studied. Some properties of anti-rectangular AG-groupoid are investigated via a concise study of its super classes. Furthermore, various conditions are explored under which an anti-rectangular AG-groupoid becomes commutative and hence a commutative semigroup. Alternatively, the paper is a contribution to the theory of commutative semigroups. Moreover, using the latest computational techniques of Prove-9 and Mace-4 a variety of examples and counterexamples are provided to depict various produced results.

1. Introduction

An AG-groupiod is usually a groupoid G, which satisfies the left invertive law, i.e., $(ab)c = (cb)a, \forall a, b, c.$ AG-groupoid was introduced by Kazim and Naseeruddin [1] in 1972 which is the generalization of a commutative semigroup. An AG-groupoid is also called an LA-semigroup, a left invertive groupoid and a right modular groupoid [1-4]. AG-groupoids have many applications in the theory of flocks, matrices, finite mathematics, fuzzy algebra and geometry [5-10]. AG-groupoids are enumerated up to order 6 [11, 12]. The data obtained is classified and as a result, many new subclasses are introduced [13-27]. Mushtaq and Khan [28] introduced some subclasses of AG-groupoids in which one class is known as anti-rectangular AG-groupoid. They investigated that anti-rectangular AG-groupoid are simple and have no proper ideals. Khan [29] studied the class of anti-rectangular AG-groupoid in detail and explored several results. It is pertinent to mention that anti-rectangular AG-groupoid are very rare as investigated by Khan [29]. It is remarkable to see that out of the total 3 AG-groupoids of order 2, only one is anti-rectangular which is associative. Similarly, out of the total 20 AG-groupoids of order 3, none is anti-rectangular. In order 4 out of the total 331 there are only 2 anti-rectangular AG-groupoids, in which one is associative and the other is non-associative and non-commutative. Previously a complete table of these AG-groupoids up to order 6 has been presented by Khan [29].

In short, there exists only one non-associative anti-rectangular AG-groupoids of order 4 and two of order 8. Similarly, the AG-groupoid of order 12 exists, but unfortunately we are unable to count how many these are. However, non-associative anti-rectangular AG-groupoids of order 2, 3, 5, 6, 7 do not exist, as discussed by Ahmad et al. [16]. It has been investigated that anti-rectangular AG-groupoid is a Latin square [30], and thus has a close relation with a group. Khan [29] explored that if simply a left identity is allowed in anti-rectangular AG-groupoid it becomes an Abelian group. By an anti-rectangular AG-groupoid, we shall mean an AG-groupoid satisfying the identity.

$$ab \cdot a = b, \forall a, b.$$

The following example constructed by the latest techniques of Mace-4 and GAP shows the existence of a non-associative anti-rectangular AG-groupoid.

Example 1: Table 1. depict a non-associative anti-rectangular AG-groupoid of order 4 and Table 2 a non-associative anti-rectangular AG-groupoid of order 8.

Table 1: Non-associative anti-rectangular AG-groupoid of order 4.

·	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

Table 2: Non-associative anti-rectangular AG-groupoid of order 8.

·	0	1	2	3	4	5	6	7
0	0	2	3	1	4	7	5	6
1	3	1	0	2	5	6	4	7
2	1	3	2	0	7	4	6	5
3	2	0	1	3	6	5	7	4
4	4	6	5	7	0	1	3	2
5	6	4	7	5	2	3	1	0
6	7	5	6	4	1	0	2	3
7	5	7	4	6	3	2	0	1

It is always of interest to study one algebraic structure for the purpose of the other to obtain better results but this cannot be conveniently done each time. However, very frequently the question arises under what conditions one structure can be obtained from the other known one. Further it also remains appealing that what is the relation among these structures and how these relations can be simpler and effective. Therefore,

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Table 3: Various AG-groupoids with their identities.

AG-groupoid	Defining identity	AG-groupoid	Defining identity
Medial	$(ab)(cd) = (ac)(bd)$	Left cancellative	$ac = ab \Rightarrow c = b$
Left nuclear square	$x^2 \cdot yz = x^2y \cdot z$	Right cancellative	$yb = zb \Rightarrow y = z$
Middle nuclear square	$xy^2 \cdot z = x \cdot y^2z$	Cancellative	Both left & right cancellative
Right nuclear square	$xy \cdot z^2 = x \cdot yz^2$	AG-3 -band	$(aa)a = a(aa) = a$
Paramedial	$(ab)(cd) = (db)(ca)$	AG-band	$aa = a$
Unipotent	$aa = bb$	$AG-3-band$	$a(aa) = (aa)a = a$
Left alternative	$(aa)b = a(ab)$	T^1 AG-groupoid	$ab = cd \Rightarrow ba = dc$
Right alternative	$(ab)b = a(bb)$	T_f^4 AG-groupoid	$ab = cd \Rightarrow ad = cb$
Flexible	$a(ba) = (ab)a$	T^2 AG-groupoid	$ab = cd \Rightarrow ac = bd$
Quasi-cancellative	$yy = yz, zz = zy$ & $yy = zy, zz = yz$ Both implies $y = z$	RAD (Right Abelian distributive)	$(ab)c = (ca)(bc)$
Transitively commutative	$ab = ba,\ bc = cb \Rightarrow ac = ca$	Left Abelian distributive	$x \cdot yz = xy \cdot zx$
Outer Repeated	$(ab)(cd) = (aa)(dd)$	Slim AG-groupoids	$a \cdot bc = ac$
Left Cheban	$a(bc \cdot d) = ca \cdot bd$	Regular AG-groupoid	Both $ca = cb$ & $ac = bc$
Right commutative	$x \cdot yz = x \cdot zy$	Locally associative	$(aa)a = a(aa)$
Weak commutative	$ab \cdot cd = dc \cdot ba$	Left transitive	$xy \cdot xz = yz$
Cyclic Associative	$a(bc) = c(ab)$	Left repeated	$wx = yz \Rightarrow ww = yy$
IR (Inner Repeated)	$(ab)(cd) = (bb)(cc)$	Right repeated	$wx = yz \Rightarrow xx = zz$
Self-dual	$a(bc) = c(ba)$	Outer dominant	$wx = yz \Rightarrow ww = zz$
Left unar	$ab = ac$	Inner dominant	$wx = yz \Rightarrow xx = yy$
Left commutative	$xy \cdot z = yx \cdot z$	Stein	$x.yz = yz.x$
Left permutable	$x \cdot yz = y \cdot xz$	Right permutable	$xy \cdot z = xz \cdot y$

using the idea of construction, we investigate that a commutative semigroup can be obtained from an anti-rectangular AG-groupoid through a variety of subclasses of AG-groupoid.

In this paper, we thoroughly study anti-rectangular AG-groupoids that satisfy the identity $ab \cdot a = b$ We investigate that an anti-rectangular AG-groupoid is a subclass of self-dual AG-groupoid, AG-3-band, locally associative AG-groupoid, flexible AG-groupoid, cancellative AG-groupoid, transitively commutative AG-groupoid, regular AG-groupoid, T_f^4-AG-groupoid, quasi-cancellative AG-groupoid, and T^3-AG-groupoid. While in Theorem 2 we prove that a commutative semigroup can be obtained from an anti-rectangular AG-groupoid when combined with any of the indicated groupoids like unipotent AG-groupoid, paramedial AG-groupoid, left transitive AG-groupoid and many more.

2. Materials and Methods

In this section, we list some of the basic definitions that will be used in the subsequent sections of this paper. We begin with the following definition.

Definition 2.1: A groupoid H is called an AG-groupoid [1, 2] if for all $a, b, c \epsilon H$ the left invertive law hold,

$(ab)c = (cb)a,$ for all a, b, c in H.

Table 3 contains various AG-groupoids that satisfy various properties and will be used in the rest of this article.

3. Results and Discussions

In this section, we find a variety of AG-groupoids for which an anti-rectangular AG-groupoid is a subclass of that AG-groupoids. First, we prove a lemma which comforts us to prove that every anti-rectangular AG-groupoid is quasi-cancellative AG-groupoid.

Lemma 1: Every AG-3-band is quasi-cancellative.

Proof: Let H be an AG-3-band. Then for all x, y in H to prove, H is quasi-cancellative we prove that H is right and left quasi-cancellative. For right quasi cancellative using medial, left invertive, locally associative laws and the assumption $xx = xy$ and $yy = yx,$ we have:

$$xx \cdot x = x \Rightarrow xy \cdot x = x \Rightarrow \big((xx \cdot x)y\big)x = x$$

$$\Rightarrow (yx \cdot xx)x = x$$

$$\Rightarrow (yy \cdot xx)x = x \Rightarrow (yx \cdot yx)x = x$$

$$\Rightarrow (yy \cdot yy)x = x$$

$$\Rightarrow (yy \cdot yy)(xx \cdot x) = x \Rightarrow (yy \cdot xx)(yy \cdot x) = x$$

$$\Rightarrow (yx \cdot yx)(yy \cdot x) = x \Rightarrow (yy \cdot yy)(yy \cdot x) = x$$

$$\Rightarrow \big((yy \cdot y)y\big)(yy \cdot x) = x \Rightarrow (yy)(yy \cdot x) = x$$

$$\Rightarrow (y \cdot yy)(yx) = x \Rightarrow y \cdot yx = x \Rightarrow y \cdot yy = x \Rightarrow y = x.$$

H is right quasi-cancellative.

Next, we prove H is left quasi-cancellative. Assume that $xx = yx$, $yy = xy$. Then

$$xx \cdot x = x \Rightarrow x \cdot xx = x \Rightarrow x \cdot yx = x$$

$$\Rightarrow (x \cdot xx)(yx) = x \Rightarrow (xy)(xx \cdot x) = x$$

$$\Rightarrow (yy)x = x \Rightarrow (xy)x = x$$

$$\Rightarrow yy \cdot x = xy \cdot x = x \Rightarrow xy \cdot y = x$$

$$\Rightarrow yy \cdot y = x \Rightarrow y = x.$$

Thus H is left quasi-cancellative. Equivalently, H is quasi-cancellative.

Remark. The converse of the above theorem is not valid as depicted in the following table of an AG-groupoid $(H, \cdot)$ that is quasi-cancellative but is not AG-3-band, as $(0 \cdot 0)0 = 0(0 \cdot 0) = 0 \cdot 1 = 3 \neq 0$.

Table 4: A quasi-cancellative but is not AG-3-band.

·	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	0	1	2	3
3	2	3	0	1

Theorem 1: Let H be an anti-rectangular AG-groupoid, then each of the following holds:

i. H is self-dual,
ii. H is AG-3-band,
iii. H is locally associative,
iv. H is flexible,
v. H is cancellative AG-groupoid,
vi. H is transitively commutative AG-groupoid,
vii. H is regular AG-groupoid,
viii. H is $T_f{}^4$- AG-groupoid,
ix. H is quasi-cancellative AG-groupoid,

Proof: Let H be an anti-rectangular AG-groupoid and $a, b, c, d \in H$.

i. Then by left invertive and medial laws, we have

$$a(bc) = (ba \cdot b)(bc) = (bc \cdot b)(ba) = c(ba).$$

Thus H is self-dual.

ii. To prove that H is AG-3-band, using part-i, self-dual, the medial, and left invertive laws we have:

$$(aa)a = (aa)(aa \cdot a) = (a \cdot aa)(aa)$$

$$= a(a(a \cdot aa)) = a(aa \cdot aa)$$

$$= a((aa \cdot a)a) = a(aa) = a$$

$$\Rightarrow (aa)a = a(aa) = a.$$

Thus H is AG-3-band.

iii. Follows by Part-ii, as

$(aa)a = a(aa)$.

Hence H is locally associative.

iv. To prove H is flexible. We use the medial, and left invertive laws and Part-iii as follows:

$$(ab)a = (ab)(aa \cdot a) = (a \cdot aa)(ba) = (aa \cdot a)(ba) = a(ba).$$

Thus H is flexible.

v. To prove H is cancellative we shall prove that H is both left cancellative and right cancellative. Then, using the anti-rectangular property we have

$$ay = az \Rightarrow (ay)a = (az)a \Rightarrow y = z.$$

Thus H is left cancellative.

For right cancellativity, assume that $ya = za$, then by Part-iv,

$$\Rightarrow a(ya) = a(za) \Rightarrow (ay)a = (az)a \Rightarrow y = z.$$

Thus H is right cancellative. Hence H is cancellative.

vi. Assume that $ab = ba$, $bc = cb$, to prove H is transitively commutative AG-groupoid, we use the assumption, part-i and part-iv to prove that $ac = ca$. Since

$$ac = (ba \cdot b)c = (b \cdot ab)c = (b \cdot ba)c = (a \cdot bb)c$$

$$= (c \cdot bb)a = (b \cdot bc)a = (b \cdot cb)a = (bc \cdot b)a = ca$$

$$\Rightarrow ac = ca.$$

Hence H is transitively commutative AG-groupoid.

vii. To prove that H is regular, we prove that H is both left and right regular. We use part-iv and part-v. To do this let

$$za = zb \Rightarrow z(ca \cdot c) = z(cb \cdot c)$$

$$\Rightarrow ca \cdot c = cb \cdot c \Rightarrow ca = cb.$$

Thus H is left regular.

Now to prove H is a right regular. Let $az = bz$

$$\Rightarrow (ca \cdot c)z = (cb.c)z \Rightarrow ca \cdot c = cb.c$$

$$\Rightarrow c \cdot ac = c \cdot bc \Rightarrow ac = bc.$$

Thus H is right regular. Hence H is regular.

viii. To prove that H is $T_f{}^4$-AG-groupoid. Assume that $ab = cd$. Then by the left invertive law, part-iv and part-v of this theorem, we have $ab = cd$

$$\Rightarrow (da \cdot d)b = (bc \cdot b)d$$

$$\Rightarrow (d \cdot ad)b = (b \cdot cb)d$$

$$\Rightarrow (b \cdot ad)d = (b \cdot cb)d$$

$$\Rightarrow b \cdot ad = b \cdot cb$$
$$\Rightarrow ad = cd.$$

Hence H is T_f^4 - AG-groupoid.

ix. By part ii. H is AG-3-band and by Lemma-1, H is quasi-cancellative AG-groupoid.

It is prominent to mention that the converse of each of the above results is not valid as can be seen in the following counterexample.

Example 2: In the tables given below, Table 5(i) is a counterexample of Theorem 1(i) of order 5, Table 5(ii) is counterexample of Theorem 1(ii) of order 3, Table 5(iii & iv) is counterexample of Theorem 1(iii & iv) of order 3, Table 5(v) is a counterexample of Theorem 1(v) of order 3 and so on.

Table 5(i): A counterexample of Theorem 1(i)

·	1	2	3	4	5
1	1	3	4	2	1
2	4	2	1	3	4
3	2	4	3	1	2
4	3	1	2	4	3
5	1	3	4	2	1

Table 5(ii): A counterexample of Theorem 1 (ii)

·	0	1	2
0	1	1	0
1	1	1	1
2	0	1	2

Table 5(iii): A counterexample of Theorem 1 (iii).

·	1	2	3
1	1	1	3
2	1	2	3
3	3	3	1

Table 5(iv): A counterexample of Theorem 1 (iv)

·	0	1	2
0	1	1	0
1	1	1	1
2	0	1	2

Table 5(v): A counterexample of Theorem 1 (v)

·	0	1	2
0	1	2	0
1	2	0	1
2	0	1	2

Table 5(vi): A counterexample of Theorem 1 (vi)

·	0	1	2	3
0	1	1	0	0
1	1	1	1	1
2	0	1	2	2
3	0	1	2	2

Table 5(vii): A counterexample of Theorem 1 (vii)

·	0	1	2	3
0	1	1	0	1
1	1	1	0	1
2	3	3	3	3
3	0	0	1	0

Table 5(viii): A counterexample of Theorem 1 (viii)

·	0	1	2	3
0	1	0	3	2
1	2	3	0	1
2	3	2	1	0
3	0	1	2	3

Table 5(ix): A counterexample of Theorem 1 (ix)

·	0	1	2	3
0	1	1	1	1
1	1	1	1	1
2	0	0	0	0
3	0	0	0	0

As discussed earlier, anti-rectangular AG-groupoid is in general is a non-associative structure. However, in the following we prove that an anti-rectangular AG-groupoid when combined with any of the following indicated groupoids, it becomes a commutative semigroup. We prove this in the form of a theorem. Furthermore, it is easy to prove that:

Proposition 1: [2] A commutative AG-groupoid is always associative.

Using this proposition we prove the following:

Theorem 2. Let H be an anti-rectangular AG-groupoid. Then H is a commutative semigroup if any of the following hold.

1. H is unipotent AG-groupoid,
2. H is paramedial AG-groupoid,
3. H is outer repeated AG-groupoid,
4. H is left transitive AG-groupoid,
5. H is left repeated AG-groupoid,
6. H is right repeated AG-groupoid,
7. H is outer dominant AG-groupoid,
8. H is inner dominant AG-groupoid,
9. H is left unar AG-groupoid,

10. H is weak commutative AG-groupoid,
11. H is left nuclear square AG-groupoid,
12. H is middle nuclear square AG-groupoid,
13. H is right nuclear square AG-groupoid,
14. H is Stein AG-groupoid,
15. H is left permutable AG-groupoid,
16. H is inner repeated AG-groupoid,
17. H is right permutable AG-groupoid,
18. H is left permutable AG-groupoid,
19. H is right alternative AG-groupoid,
20. H is left alternative AG-groupoid,
21. H is left abelian distributive AG-groupoid,
22. H is right abelian distributive AG-groupoid,
23. H is right commutative AG-groupoid,
24. H is left commutative AG-groupoid,
25. H is slim AG-groupoid.

Proof: Let *H* be an anti-rectangular AG-groupoid and *w*, *x*, *y*, *z* be elements in *H*. Then

1. By the assumption and using Parts i, and ii of Theorem 1 we have,
$$xy = x(yy \cdot y) = y(yy \cdot x) = y(xx \cdot x) = yx.$$
2. By the assumption and using Parts i, and ii of Theorem 1 and the left invertive law we have,
$$xy = (xx \cdot x)y = yx \cdot xx = xx \cdot xy =$$
$$= y \cdot x(xx) = yx.$$
3. By the assumption and using Parts i, and ii of Theorem 1 we have,
$$xy = (yx \cdot y)y = (yy)(yx) = yy \cdot xx =$$
$$= y(yy) \cdot (xx)x = yx.$$
4. By the assumption and using Part 1 of Theorem 2 we have,
$$xy = (yx \cdot y)y = yy \cdot yx = yx.$$
5. By the assumption and using Parts i, ii, and iv of Theorem 1 we have,
$$xy = x(yy \cdot y) = y(yy \cdot x) = xx = yy$$
$$= xy = y\big((yx)y\big) = \big(y(yx)\big)y = yx.$$
6. Let $wx = zy$ this implies
$$xx = yy \implies yx = xy.$$
7. Let $xw = zy$ this implies
$$xx = yy \implies xy = y(yx \cdot y) = yx.$$
8. Assume $xw = zy$ this implies
$$ww = zz \implies zw = z(zw \cdot z) = zw.$$
9. By the assumption and using Parts i, and ii of Theorem 1 we have,
$$xy = x(yy \cdot y) = y(yy \cdot x) = yx.$$
10. By the assumption and using Parts i, and ii of Theorem 1 and the left invertive law we have,
$$xy = (yx \cdot y)y = yy \cdot yx = xy \cdot yy = (yy \cdot y)x = yx.$$
11. By the assumption and using Part i of Theorem 1 and the left invertive law we have,
$$xy = (xx \cdot x)y = xx \cdot xy = (xy \cdot x)x = yx.$$
12. By the assumption and using Part ii of Theorem 1 and the left invertive law we have,
$$xy = x(yy \cdot y) = (x \cdot yy)y = (y \cdot yy)x = yx.$$
13. By the assumption and using Part ii, of Theorem 1 and the left invertive law we have,
$$xy = x(y \cdot yy) = xy \cdot yy = (yy \cdot y)x = yx.$$
14. By the assumption and using Parts i, and iv of Theorem 1 and the left invertive law we have,
$$xy = (yx \cdot y)y = (y \cdot xy)y = (xy \cdot y)y = (yy)(xy)$$
$$= y(x \cdot yy) = y(y \cdot yx) = y(yx \cdot y) = yx$$
15. By the assumption and using Part ii of Theorem 1 and the medial law we have,
$$xy = x(xy \cdot x) = xy \cdot xx = xx \cdot yx = y(xx \cdot x) = yx.$$
16. By the assumption and using Parts i, ii, and iv of Theorem 1 we have,
$$xy = x(xy \cdot x) = x(x \cdot yx) = yx \cdot xx =$$
$$xx \cdot xx = (xy)x \cdot x(xx) = yx.$$
17. By the assumption and using Part ii of Theorem 1 we have,
$$xy = x(yy \cdot y) = y(yx \cdot y) = yx.$$
18. By the assumption and using Part 2 of Theorem 2 we have,
$$xy = (xx \cdot x)y = (xy \cdot x)x = yx.$$
19. By the assumption and using part ii of Theorem 1, and the medial and left invertive laws we have,
$$xy = (yx \cdot y)y = yx \cdot yy = yy \cdot xy = \big((xy)y\big)y$$
$$= xy \cdot yy = \big((yy)y\big) = yx.$$
20. By the assumption and using Parts i, ii, and iv of Theorem 1 we have,
$$xy = (yx \cdot y)y = yy \cdot yx = y\big(y(yx)\big) = y\big(x(yy)\big) =$$
$$= yy \cdot xy = y\big(y(xy)\big) = y\big((yx)y\big) = yx.$$
21. By the assumption and using Parts i, ii, and iv of Theorem 1 and the medial law we have,
$$xy = (yx \cdot y)y = (y \cdot xy)y = (yx \cdot yy)y = (yy \cdot xy)y$$
$$= (y \cdot yx)y = (x \cdot yy)y = (y \cdot yy)x = yx.$$
22. By the assumption and using Parts i, ii, and iv of Theorem 1 and the left invertive law we have,
$$xy = x(yy \cdot y) = y(yy \cdot x) = y(xy \cdot yx) =$$

$$= y((yx)y \cdot x) = y(xx) = y(x \cdot (x \cdot xx))$$
$$= y(xx \cdot xx) = y(xx \cdot x) = yx.$$

23. By the assumption and using Parts i, ii, and iv of Theorem 1 and the left invertive law we have,

$$xy = (x \cdot xx)y = (y \cdot xx)x = (x \cdot xy)x =$$
$$= (x \cdot yx)x = (xy \cdot x)x = yx.$$

24. By the assumption and using Parts i, ii, and iv of Theorem 1 and the left invertive law we have,

$$xy = x(yy \cdot y) = y(yy \cdot x) = y(xy \cdot y) =$$
$$= y(yx \cdot y) = yx.$$

25. By using Part i. of Theorem 1and the slim law we have,

$$xy = x(zy) = y(zx) = yx.$$

4. Conclusions

In this paper, we have investigated various classes that properly contain the class of anti-rectangular AG-groups. We have also mentioned twenty five conditions for which an anti-rectangular AG-groupoid becomes a commutative semigroup. This presents an elegant relationship among AG-groupoid and a commutative semigroup. It is interesting to mention that finite semigroups are very rare in the literature. The construction of commutative semigroup introduced here in this article thus contributes to the theory of semigroups in general and to the commutative semigroup in special. It is worth to mention that all the examples and counterexamples are constructed with the help of modern computational techniques of Mace-4 and GAP that proves authenticity of the investigated results.

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