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Calculation of Solar Trajectory in the Sky and Solar Analemma as Observed from the Earth

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ABSTRACT

In a previous companion paper "On the elliptical orbit of the Earth and the position of the Sun in the sky: an engineering approach," published in The NUCLEUS, we presented various computational methodologies for the position/trajectory of the Sun in the sky of an observer at Earth [1]. In this paper, the methodology for calculation of solar analemma (as observed from the earth surface) has been presented, along with an elaboration of the "Equation of Time," as called in literature [2,3]. The computational methodologies presented in the earlier paper included: 1) an analytical approach; 2) a numerical algorithm; and 3) a Solar Position Algorithm commonly abbreviated as PSA from the Spanish name of its developer Plataforma Solar de Almería [4]. In the numerical approach, Earth's momentum equation written in a polar coordinate system (r, θ) was numerically solved. It was also demonstrated that if the Earth's momentum equation was transformed to eliminate the time dependence, it could be solved analytically. In this paper, a Cartesian coordinate system is used to calculate the coordinates of the pole star (Polaris) and its declination angle. The position vector of an observer that rotates with the Earth is calculated using a new Cartesian coordinate system, whose origin is located at the center of the Earth. The solar elevation angle and the solar azimuth angle are obtained by performing a set of rotations of this new coordinate system. Towards the end, the Equation of Time (EOT) is explained and used for calculating the solar analemma.

Keywords: *Sun trajectory; Analemma; Solar declination angle, Solar azimuth angle; Equation of Time*

1. Introduction

The variation in the position of Sun in the sky over an observer is a natural phenomenon that has intrigued humankind since forever. The position of Sun has been correlated with the occurrence of natural phenomena (volcanic activity, storm cycles, earthquakes, etc.) [5]. The motion of the Sun has also been considered as a measure of time, or as a phenomenon that governs the agricultural yields/cycles and outbreak of pandemics [6]. Due to the increasing price of petroleum, engineers need to efficiently extract energy from renewable sources, such as winds and the Sun. If civil engineers efficiently use the solar energy, they may design reliable intelligent buildings and sustainable environments. In the near future, the task of the engineers will attain more importance; however, sometimes they do not have enough understanding of the mathematical techniques that the astronomers use to calculate the Sun's position in the sky. In this paper an attempt has been made to bridge the gap.

In some recently published papers, simplified techniques have been presented to calculate the Sun's position in the sky, however, it was pointed out that there was a need to develop and report new mathematical algorithms, suitable for amateur astronomers, students and practitioners in the field of solar energy, see for instance [7]. A simple parametric model, that describes the basic principles of the visible Sun path on the celestial sphere, has been presented in an earlier paper [8]. A review of the Sun's position algorithms has been published by Blanco et. al [9].

The Sun position algorithms are sophisticated schemes, which compute the position of the Sun in ecliptic, celestial and horizontal coordinates. Very recently, a review of the Sun position algorithms has been presented in [10]. On the internet it is also possible to find and to execute computer codes to calculate the position of the Sun in the sky, see for instance [11]. The purpose of this paper and the companion paper [1] is to present a self-contained material suitable for energy engineers to determine the Sun's position in the sky.

In section 2, methodology for the solar elevation angle, and the solar azimuth angle measured from north are presented. The Equation of Time (EOT) with its explanation is presented in section 3. Section 4 shows the calculation of analemma obtained using EOT [12]. It may be mentioned that an analemma is a diagram showing the position of the Sun in the sky as seen from a fixed location on Earth at the same mean solar time [2], and very interestingly resembles to the shape of figure 8 (eight) [3]. The results and discussion are presented in section 5.

2. Calculation of the Solar Elevation Angle and the Solar Azimuth Angle

Before presenting the calculation of solar elevation and azimuthal angles, it may be recalled from the basic geometry that the rotations of the Earth are: (i) about its axis that points towards the North star, and (ii) around the Sun in an elliptical trajectory. A fixed Cartesian coordinate system (o, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$) could be defined, whose origin is located at the center of the Earth (see Fig. 1). Its plane $\hat{x}_1$- $\hat{x}_2$ is on the Earth's equatorial plane, and (a) the position vector of the Sun moves on the plane $\hat{x}_1$- $\hat{x}_3$, (b) its $\hat{x}_3$ axis points towards the star Polaris. The orientation of its $\hat{x}_1$, axis is defined together with the initial value (at $t^* = 0$) of the rotation angle ρ. In this paper, we have assumed that at $t^* = 0$, $\rho = 0$ radians. We can further define two vectors, the vector $\hat{x}_{obs}(t^*)$, which is the position vector of an observer that is located at a certain fixed latitude d on the Earth's surface, and the vector $\hat{x}^*_{Sun}(t^*)$, which is the Sun's position vector. Notice that the vector $\hat{x}_{obs}(t^*)$ rotates at the same angular velocity as the Earth. In the model, it is assumed that the Earth's rotation angle ρ is $0 \leq \rho \leq 2\pi$, where 2π radian corresponds to one day (24 hours or 86400 seconds), and the

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dimensionless radius of the Earth is equal to one [1]. The increment of the rotation angle Δρ (which corresponds to the time step Δt = 60 s of the numerical solution) is calculated as:

$$\Delta\rho = \frac{2\pi \text{ x } 60}{24 \text{ x } 3600} = 0.00436 \text{ radians} \quad (1)$$

The dimensionless three components of the rotating vector $\hat{x}_{obs}(t^*)$ referred to the fixed Cartesian coordinate system (o, $\hat{x}$) are given as:

$$\hat{x}^*_{1\text{ obs}}(t^*) = \cos\delta\cos\rho(t^*)\,,\ \hat{x}^*_{2\text{ obs}}(t^*) = \cos\delta\sin\rho(t^*)\,,$$
$$\hat{x}^*_{3\text{ obs}}(t^*) = \sin\delta \quad (2a)$$

and the dimensionless components of the Sun's position vector $\hat{x}^*_{Sun}(t^*)$, which oscillates from $\beta(t^*) = -23.45^o$ to $\beta(t^*) = 23.45^o$ on the plane $\hat{x}_1$- $\hat{x}_3$, are the following:

$$\hat{x}^*_{1\text{sun}}(t^*) = \cos\beta(t^*)\,,\hat{x}^*_{2\text{ sun}}(t^*) = 0, \hat{x}^*_{3\text{ sun}}(t^*) = \sin\beta(t^*) \quad (2b)$$

2.1 Solar elevation angle

Solar elevation angle is the value that describes how high the Sun is. In order to calculate the solar elevation angle α_e, we transform (at each time step Δt*) the dimensionless components of the vector $\hat{x}^*_{obs}(t^*)$, see Eq. (2a), into a new Cartesian coordinate system (o, $\hat{x}'_1$, $\hat{x}'_2$, $\hat{x}'_3$) which has the following characteristics (see Fig. 1): (i) its origin is located at the center of the Earth, (ii) its axis $\hat{x}'_1$ always points towards the Sun (that moves on the plane $\hat{x}_1$ - $\hat{x}_3$), (iii) its axis $\hat{x}'_2$ always coincides with the axis $\hat{x}_2$ of the fixed Cartesian coordinate system (o, $\hat{x}$), (iv) it oscillates (from $\beta(t^*) = -23.45^o$ to $\beta(t^*) = 23.45^o$) about its axis $\hat{x}'_2$. The dimensionless components of the vector $\hat{x}'^*_{obs}(t^*)$ are obtained by using the transformation rule explained below.

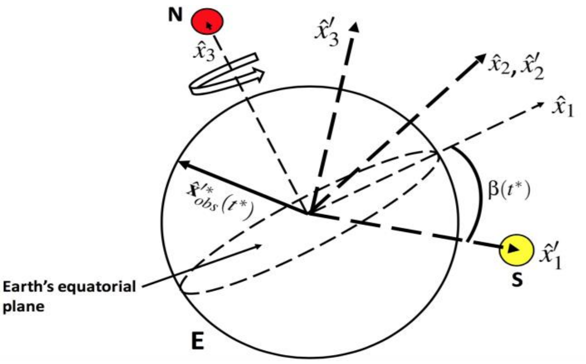

Fig. 1 Cartesian coordinate system (o, $\hat{x}'_1$, $\hat{x}'_2$, $\hat{x}'_3$), whose origin is located at the center of the Earth. Its axis $\hat{x}'_1$ always points towards the Sun. Its axis $\hat{x}'_2$ always coincides with the axis $\hat{x}_2$ of the fixed Cartesian coordinate system o, $\hat{x}$. It oscillates, from $\beta(t^*) = -23.45^o$ to $\beta(t^*) = +23.45^o$, about its axis $\hat{x}'_2$.

$$\begin{bmatrix} \hat{x}'^*_{1\text{ obs}}(t^*) \\ \hat{x}'^*_{2\text{ obs}}(t^*) \\ \hat{x}'^*_{3\text{ obs}}(t^*) \end{bmatrix} =$$
$$\begin{bmatrix} \cos\beta(t^*) & 0 & \cos(90^o - \beta(t^*)) \\ 0 & 1 & 0 \\ \cos(90^\circ + \beta(t^*)) & 0 & \cos\beta(t^*) \end{bmatrix} \begin{bmatrix} \hat{x}^*_{1\text{ obs}}(t^*) \\ \hat{x}^*_{2\text{ obs}}(t^*) \\ \hat{x}^*_{3\text{ obs}}(t^*) \end{bmatrix} \quad (3)$$

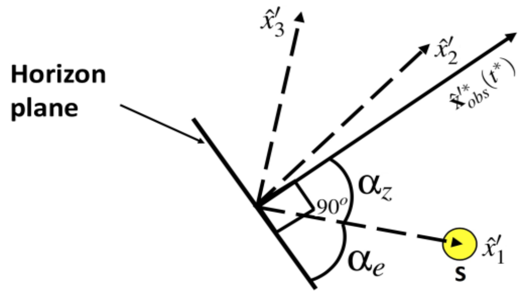

Fig. 2 The solar zenith angle α_z and the solar elevation angle α_e.

In vector form and using Eq. (2a), the Eq. (3) can be written as:

$$\hat{x}'^*_{obs}(t^*) = \left(\cos\beta(t^*)\cos\delta\cos\rho(t^*) + \cos(90^\circ - \beta(t^*))\sin\delta\right)\hat{\imath}'_1 + \cos\delta\sin\rho(t^*)\hat{\imath}'_2$$
$$+\left(\cos(90^o + \beta(t^*))\cos\delta\cos\rho(t^*) + \cos\beta(t^*)\sin\delta\right)\hat{\mathbf{i}}'_3 \quad (4)$$

The dot product between the unit position vector of the observer $\hat{x}'^*_{obs}(t^*)$ and the unit vector $\hat{\imath}'_1$ that points towards the Sun, is obtained as (see Fig. 2):

$$\hat{x}'^*_{obs}(t^*)\cdot\hat{\imath}'_1 = |\hat{x}'^*_{obs}(t^*)||\hat{\imath}'_1|\cos\alpha_z$$
$$= \cos\beta(t^*)\cos\delta\cos\rho(t^*) + \sin\beta(t^*)\sin\delta \quad (5)$$

where, the trigonometric identity $\cos(90^\circ - \beta(t^*)) = \sin\beta(t^*)$ has been used. From Eq. (5) the solar zenith angle α_Z is obtained as:

$$\alpha_z = \cos^{-1}[\cos\beta(t^*)\cos\delta\cos\rho(t^*) + \sin\beta(t^*)\sin\delta] \quad (6)$$

The solar elevation angle α_e is obtained as:

$$\alpha_e = 90^\circ - \alpha_z \quad (7)$$

2.2 The solar azimuth angle measured from north

The solar azimuth angle is the value that describes in which direction of the Sun is from north of the observer's horizon plane. The solar azimuth angle is obtained by performing two rotations of the Cartesian coordinate system (o, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$). After the two rotations (represented by the transformation matrices A and B) have been carried out (Fig.3) the unit vector $\bar{\bar{\hat{\imath}}}_3$ of the new Cartesian coordinate

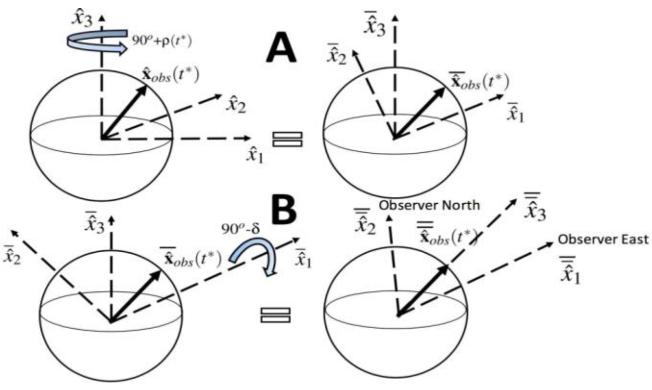

Fig. 3 Transformation matrices A and B from the Cartesian coordinate system o, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ to the Cartesian coordinate system o, $\bar{\bar{\hat{x}}}_1$ $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$.

system (o, $\bar{\bar{\hat{x}}}_1$ $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$) coincides with the position vector of the observer $\hat{x}^*_{obs}(t^*)$ see Eq. (2a), while the units vectors $\bar{\bar{\hat{\imath}}}_1$ and $\bar{\bar{\hat{\imath}}}_2$ are directed to the east and north directions, respectively, of the observer. Note that the plane defined by the two unit vectors $\bar{\bar{\hat{\imath}}}_1$ and $\bar{\bar{\hat{\imath}}}_2$ is (in a natural way) the horizon plane of the observer.

By performing this procedure, the Sun's position vector $\hat{x}^*_{Sun}(t^*)$, see Eq. (2b), that is defined in the coordinate system (o, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$), is transformed into the coordinate system (o, $\bar{\bar{\hat{x}}}_1$ $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$). The angle between the unit vector, $\bar{\bar{\hat{\imath}}}_2$, that is directed to north of the observer, and the projection of the vector $\bar{\bar{\hat{x}}}^*_{Sun}(t^*)$, on the plane $\bar{\bar{\hat{\imath}}}_1$ - $\bar{\bar{\hat{\imath}}}_2$ is the solar azimuth angle, α_a.

2.2.1 *The transformation matrices A and B*

The first rotation is about the axis $\hat{x}_3$ of the coordinate system (o, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$). If the rotation angle is 90º + ρ(t*), the first transformation matrix A is as follows:

$$\mathrm{A} = \begin{bmatrix} -\sin\rho(t^*) & \cos\rho(t^*) & 0 \\ -\cos\rho(t^*) & -\sin\rho(t^*) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

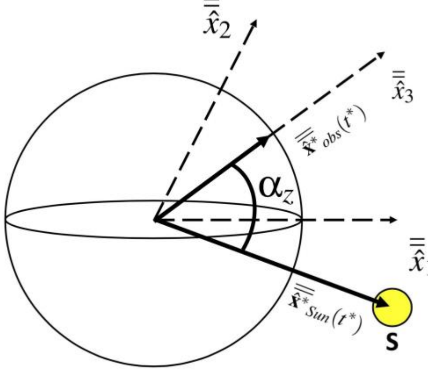

Fig. 4 The observer position vector $\bar{\bar{\hat{x}}}^*_{obs}(t^*)$, the Sun position vector $\bar{\bar{\hat{x}}}^*_{Sun}(t^*)$ and the solar zenith angle α_z in the Cartesian coordinate system o, $\bar{\bar{\hat{x}}}_1$, $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$.

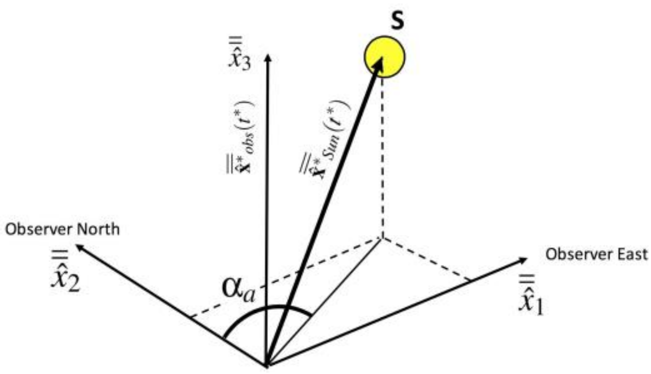

Fig. 5 The observer position vector $\bar{\bar{\hat{x}}}^*_{obs}(t^*)$, the Sun position vector $\bar{\bar{\hat{x}}}^*_{sun}(t^*)$ and the solar azimuth angle measured from north α_a in the Cartesian coordinate system o, $\bar{\bar{\hat{x}}}_1$, $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$. The observer horizon plane is defined by the plane $\bar{\bar{\hat{x}}}_1$ - $\bar{\bar{\hat{x}}}_2$, where $\bar{\bar{\hat{x}}}_1$ is the observer's East while $\bar{\bar{\hat{x}}}_2$ is the observer North.

After the first rotation, the Cartesian coordinate system (o, $\overline{\hat{x}_1}$, $\overline{\hat{x}_2}$, $\overline{\hat{x}_3}$) is obtained (top panel of Fig. 3). The second rotation is about the axis $\overline{\hat{x}_1}$ of the new coordinate system. If the rotation angle is 90º - δ, the second transformation matrix B is defined as follows:

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\delta & \cos\delta \\ 0 & -\cos\delta & \sin\delta \end{bmatrix} \quad (9)$$

After the second rotation, the Cartesian coordinate system (o, $\bar{\bar{\hat{x}}}_1$, $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$) is obtained, see bottom panel of Fig. 3. Now, the axis $\bar{\bar{\hat{x}}}_3$ coincides with the position vector of the observer $\hat{x}^*_{obs}(t^*)$, and the axes $\bar{\bar{\hat{x}}}_1$ and $\bar{\bar{\hat{x}}}_2$ define the observer's horizon plane. The axis $\bar{\bar{\hat{x}}}_1$ is now directed towards the east, while the axis $\bar{\bar{\hat{x}}}_2$ is directed to the north of the horizon plane.

The global transformation matrix from the coordinate system (o, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$) to the coordinate system (o, $\bar{\bar{\hat{x}}}_1$, $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$) is defined as R = BA, which is written as:

$$\mathrm{R} = \mathrm{BA} = \begin{bmatrix} -\sin\rho(t^*) & \cos\rho(t^*) & 0 \\ -\sin\delta\cos\rho(t^*) & -\sin\delta\sin\rho(t^*) & \cos\delta \\ \cos\delta\cos\rho(t^*) & \cos\delta\sin\rho(t^*) & \sin\delta \end{bmatrix} \quad (10)$$

The next step is to transform the Sun's position vector $\hat{x}^*_{Sun}(t^*)$, see Eq. (2b), into the new coordinate system (o, $\bar{\bar{\hat{x}}}_1$, $\bar{\bar{\hat{x}}}_2$, $\bar{\bar{\hat{x}}}_3$), hence the following transformation is carried out:

$$\bar{\bar{\hat{x}}}_{\mathrm{Sun}}(t^*) = \begin{bmatrix} -\sin\rho(t^*) & \cos\rho(t^*) & 0 \\ -\sin\delta\cos\rho(t^*) & -\sin\delta\sin\rho(t^*) & \cos\delta \\ \cos\delta\cos\rho(t^*) & \cos\delta\sin\rho(t^*) & \sin\delta \end{bmatrix} \begin{bmatrix} \cos\beta(t^*) \\ 0 \\ \sin\beta(t^*) \end{bmatrix} \quad (11)$$

or,

$$\bar{\bar{\hat{x}}}_{\mathrm{Sun}}(t^*) = \begin{bmatrix} -\sin\rho(t^*)\cos\beta(t^*) \\ -\sin\delta\cos\rho(t^*)\cos\beta(t^*) + \cos\delta\sin\beta(t^*) \\ \cos\delta\cos\rho(t^*)\cos\beta(t^*) + \sin\delta\sin\beta(t^*) \end{bmatrix} \quad (12)$$

From the three components of the vector $\bar{\bar{\hat{x}}}^*_{Sun}(t^*)$, the solar zenith angle, α_z (hence the elevation angle α_e), see Fig. 4, and the solar azimuth angle measured from north of the observer's horizon (see Fig. 5) are obtained.

The third direction cosine of the vector $\bar{\bar{\hat{x}}}^*_{Sun}(t^*)$ see Eq. (12), represents the cosine of the angle between the position vector of the observer and the Sun's position vector, that is, the cosine of the zenith angle α_z, which has been previously evaluated, see Eq. (6). The solar azimuth angle measured from north of the observer's horizon is obtained by taking the tangent of the angle measured from the axis $\bar{\bar{\hat{x}}}_2$ (north direction) to the projection of the vector $\bar{\bar{\hat{\mathbf{x}}}}^*_{Sun}(t^*)$ on the plane $\bar{\bar{\hat{x}}}_1$ - $\bar{\bar{\hat{x}}}_2$ that is:

$$\tan\alpha_a = \frac{\bar{\bar{\hat{\mathbf{x}}}}^*_{1\,Sun}(t^*)}{\bar{\bar{\hat{\mathbf{x}}}}^*_{2\,Sun}(t^*)} \quad (13)$$

hence, the solar azimuth angle measured from north (i.e. α_a), see Eq. (12), is given as

$$\alpha_a = \tan^{-1}\left\{\frac{-\sin\rho(t^*)\cos\beta(t^*)}{\cos\delta\sin\beta(t^*) - \sin\delta\cos\rho(t^*)\cos\beta(t^*)}\right\} \quad (14)$$

In Fig. 6, a two dimensional map is showing the solar azimuth angle α_a at the abscissa and the solar elevation angle, α_e at the ordinate, for four days (March 20, June 21, September 19 and December 21) of the year 2013, corresponding to a latitude $\delta = 40.73^o$. The declination angle (β) is used to calculate the angles α_a and α_e. The declination angle (β) at the beginning of the day and at the end of the day (see the values of Declin. start and Declin. end), as well as the sunrise and the sunset regions, are displayed in each panel of Figure. Notice that for the latitude $\delta = 40.73^o$ (and for the locations on the Earth's northern hemisphere with latitude greater than 23.45^o) the Sun is always south of the place (see the values of Declin. start and Declin. end in Fig. 6), hence the solar azimuth angle α_a measured from north of the observer's horizon plane at noon is always $\alpha_a = 180^o$ (south direction of the observer's horizon plane). In Fig. 6, it is observed that the sunrise region is located in the interval $0^o < \alpha_a < 180^o$ (north-east-south direction of the observer's horizon plane), while the sunset region is in the interval between $180^o < \alpha_a < 360^o$ (south-west-north direction of the observer's horizon plane). The elevation angle α_e ($\alpha_e \geq 0^o$) is measured from the location at which α_a is evaluated. Then in Fig. 6, if α_a is lower than 90^o or if α_a is greater than 270^o, the Sun is north of the observer's horizon plane and α_e is measured from north of this plane. On the other hand, if α_a is in the interval $90^o < \alpha_a < 270^o$ the Sun is south of the observer's horizon plane, then α_e is measured from south of this plane. However, for the locations on the Earth's north hemisphere with latitude $\delta < 23.45^o$ (not shown here), in some dates of the year, the Sun is north of the place, hence the physical interpretation of the angles α_e and α_a is different. That is, when the Sun is north of the place, the solar azimuth angle measured from north of the observer's horizon plane is in the interval $-90^o < \alpha_a < 90^o$ degrees. Where, the interval from 90^o to 0^o corresponds to the region from sunrise to noon, while the interval from 0^o to -90^o corresponds to the region from noon to sunset.

3. Equation of Time

The results of the Sun's position in the sky of an observer, presented in the previous sections, are not fully correlated with the observations of one person that attempts to see the position of the Sun at the same clock time every day along the year. This observer will notice that the Sun is not at the same position in the sky. The reason is that the clock time is based on a standard measure that considers a fictitious Earth that travels around the Sun in a circular trajectory with constant tangential velocity (hence with constant angular velocity). Additionally, the standard measure of time (clock time) takes into account that the Sun is always located at the Earth's equator. As it was previously shown, the trajectory of the true Earth is elliptical, and its tangential velocity v_t as well as the

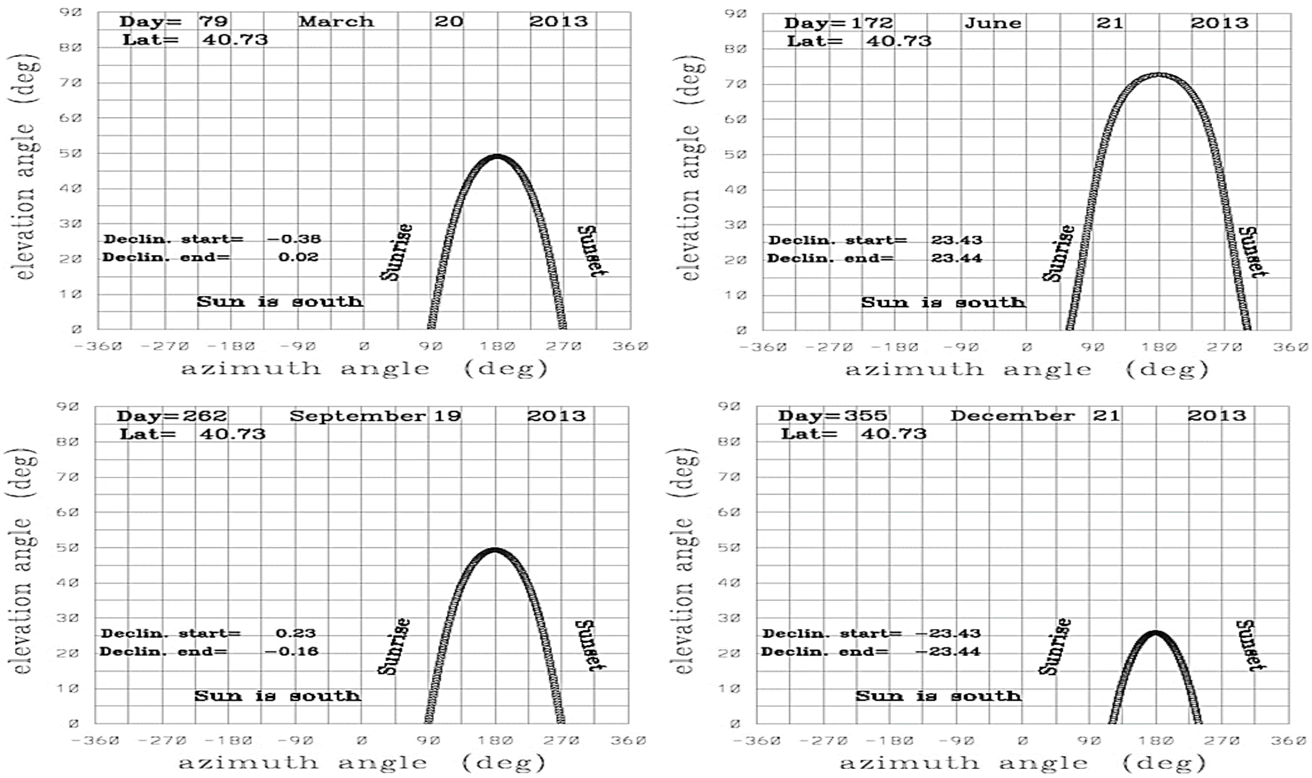

Fig. 6 Map of the solar azimuth angle is measured from north of the observer's horizon plane α_a (see Eq. 14) and the solar elevation angle α_e is calculated from Eq. (7). The observer is located at a latitude $\delta = 40.73^o$. Four days of the year 2013 are shown. The number of the day along the year, the sunrise, and the sunset regions are displayed. The declination angle β at the beginning of the day in degrees (Declin. start) and the declination angle β at the end of the day in degrees (Declin. end) are also displayed. The legend Sun is south, means that the declination angle β is lower than the latitude of the place $\delta = 40.73^o$.

Sun's position relative to the Earth's equator (the declination angle β), are both functions of time. The difference between the standard measure of time (clock time) and the Sun's position in the sky of an observer (solar time) is known as the equation of time or also called correction of time.

3.1 Standard measure of time (clock time)

In the standard measure of time model, it is assumed that a fictitious Earth with constant angular velocity Ω_{Circ} follows a circular trajectory. The angular velocity Ω_{Circ} is given by:

$$\Omega_{Circ} = 360/365 = 0.98630136 \text{ [Degrees/day]} \quad (15a)$$

or,

$$\Omega_{Circ} = 360/365 \times 1/24 \times 1/60 = 0.0006849315 \text{ [Degrees/min]} \quad (15b)$$

where, [Degrees] is the unit of angle measured along the circular trajectory.

In the standard measure of time, it is also assumed that the fictitious Earth's angular velocity ω_{Circ} about its rotation axis, which is normal to the ecliptic plane, is 360 degrees per day, or:

$$\omega_{Circ} = 360/24 \times 1/60 = 0.25 \text{ [degrees/min]} \quad (16)$$

where, [degrees] is the unit of angle of the fictitious Earth's rotation about its axis. In the clock time model, it is assumed that the position of the Sun relative to the position of an observer is exactly the same each 24 hours (i.e. each 360 degrees or each 0.98630136 Degrees). The Sun's position relative to the observer in the circular trajectory is evaluated by considering that the fictitious Earth travels on a circle with unit radius *R*. At the center of the circle is the Sun. As shown in Fig. 7, the origin of the fixed coordinate system (O, X_1, X_2) is located at the center of the circle (at the Sun's position), while the origin of the moving and locally rotating coordinate system (o, x_1, x_2) is located at the Earth's position $X_E(t)$.

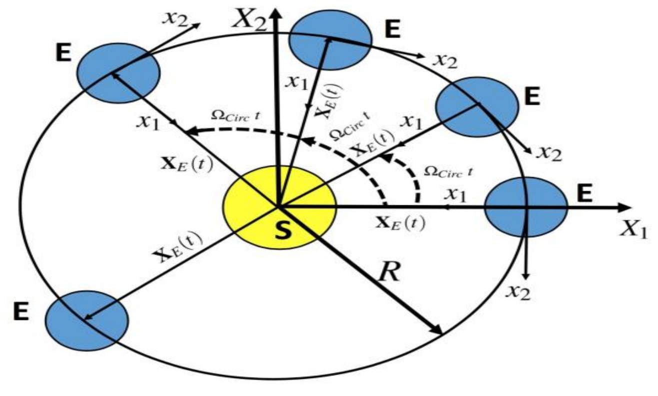

Fig. 7 Circular trajectory of the fictitious Earth: two Cartesian coordinate systems are defined. The fixed coordinate system (O, X_1, X_2) whose origin is located at the center of the circle with unit radius R (at the Sun's position), and the moving and rotating coordinate system (o, x_1, x_2), whose origin is located along the circle, i.e. at the Earth's position vector $X_E(t) = R\cos(\Omega_{Circ}\, t)\, I_1 + R\sin(\Omega_{Circ}\, t)\, I_2$. The axis x_1 always is directed to the Sun, hence it is in the same direction of the vector $X_E(t)$.

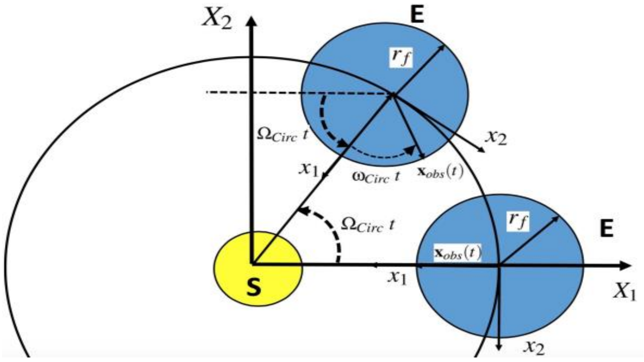

Fig. 8 Circular trajectory of the fictitious Earth: the moving coordinate system (o, x_1, x_2), rotates by an angle Ω_{Circ} t. In the moving coordinate system, the position of an observer on the fictitious Earth surface is given as $x_{obs}(t) = r_f \cos((\omega_{Circ} + \Omega_{Circ})t)\, i_1 + r_f \sin((\omega_{Circ} + \Omega_{Circ})t)\, i_2$, where r_f is the unit radius of the fictitious Earth.

In the fixed coordinate system the position vector of the Earth is defined as:

$$X_E(t) = R\cos\left(\Omega_{Circ}\, t\right) I_1 + R\sin\left(\Omega_{Circ}\, t\right) I_2, \quad (17)$$

where, I_1 and I_2 are the unit vectors of the fixed coordinate system and t is the elapsed time in minutes. In the fictitious Earth model, it is assumed that the coordinate system (o, x_1, x_2) is also rotating an angle Ω_{Circ} t, in such a way that the axis x_1 is always in the direction of the vector $X_E(t)$, see Figs. 7 and 8. In the moving and rotating coordinate system, the position of an observer on the fictitious Earth' surface is given as:

$$x_{obs}(t) = r_f \cos\left((\omega_{Circ} + \Omega_{Circ})t\right) i_1 + r_f \sin\left((\omega_{Circ} + \Omega_{Circ})t\right) i_2 \quad (18)$$

where, i_1 and i_2 are the unit vectors of the moving and rotating coordinate system and r_f is the unit radius of the fictitious Earth. In the fixed coordinate system (O, X_1, X_2), the position vector $X_{obs}(t)$ of the observer can be evaluated through the use of the two components of the position vector $x_{obs}(t)$ (see Eq. 18), which are defined on the surface of the fictitious Earth and in the coordinate system (o, x_1, x_2). The position vector $X_{obs}(t) = X_E(t) - x_{obs}(t)$ (see Fig. 8), is calculated as:

$$X_{obs}(t) = \left(R\cos(\Omega_{Circ}\, t) - r_f\cos\left((\omega_{Circ} + \Omega_{Circ})t\right)\right) I_1 + \left(R\sin(\Omega_{Circ}\, t) - r_f\sin\left((\omega_{Circ} + \Omega_{Circ})t\right)\right) I_2 \quad (19)$$

Now, in the fixed Cartesian coordinate system we have the following three vectors: $X_E(t)$, $X_{obs}(t)$ and X_{E-obs}. The last vector is the relative vector from the center of the fictitious Earth to the observer. The relationship between the three vectors (that is $X_{E-obs}(t) = X_{obs}(t) - X_E(t)$), is given as follows:

$$X_{E-obs}(t) = -r_f\cos\left((\omega_{Circ} + \Omega_{Circ})t\right) I_1 - r_f\sin\left((\omega_{Circ} + \Omega_{Circ})t\right) I_2 \quad (20)$$

3.2 First component of the equation of time: correction due to eccentricity of the Earth's orbit

It is clear from the previous discussion that the true Earth (that moves in an elliptical trajectory) has an angular velocity ($\Omega_{Ell}(t) = d\theta(t)/dt$), that is not a constant. As presented earlier, two Cartesian coordinate systems are defined to represent the elliptical path of the true Earth. It is assumed that the rotation axis of the true Earth is normal to the ecliptic plane. The origin of the fixed coordinate system (O, X_1, X_2) is located at the focus of the ellipse (at the Sun's position), while the origin of the moving coordinate system (o, x_1, x_2) is located at the center of the Earth that is moving along its elliptical orbit. In the fixed coordinate system the position vector of the true Earth is given as:

$$\mathrm{X}_E(t) = r^*(t)\cos\theta(t)\mathrm{I}_1 + r^*(t)\sin\theta(t)\mathrm{I}_2, \quad (21)$$

where, r*(t) is the dimensionless polar coordinate. If we assume that the moving coordinate system also rotates as a function of time (i.e the angle θ(t)), it is explicitly considered that the axis x_1 is always in the direction of the vector $X_E(t)$, then the position of the observer on the surface of the true Earth is given as:

$$\mathrm{x}_{\mathrm{obs}}(t) = r_t\cos(\omega_{\mathrm{Circ}}\,t + \theta(t))\mathrm{i}_1 + r_t\sin(\omega_{\mathrm{Circ}}\,t + \theta(t))\mathrm{i}_2, \quad (22)$$

where, r_t is the unit radius of the true Earth. In the fixed coordinate system (O, X_1, X_2), the coordinates of the observer, which are defined in the moving coordinate system (see Eq. 22), are obtained as:

$$\begin{aligned}\mathrm{X}_{\mathrm{obs}}(t) &= \mathrm{X}_E(t) - \mathrm{x}_{\mathrm{obs}}(t)\\ &= \left(r^*(t)\cos\theta(t) - r_t\cos(\omega_{\mathrm{Circ}}\,t + \theta(t))\right)\mathrm{I}_1 + \\ &\quad \left(r^*(t)\sin\theta(t) - r_t\sin(\omega_{\mathrm{Circ}}\,t + \theta(t))\right)\mathrm{I}_2.\end{aligned} \quad (23)$$

In the fixed Cartesian coordinate system we have the three vectors: $X_E(t)$, $X_{obs}(t)$ and X_{E-obs}. It is worth mentioning that the graphs, as a function of time, of the vectors defined by Eqs. (21)-(23), confirm again the fact that the observer (even if the trajectory is elliptical with variable angular velocity $\Omega_{Ell}(t)$) sees, along the year, the Sun at the same position at the same hour of the day, as it was the case of the circular trajectory with constant angular velocity Ω_{circ}. The reason of this behavior of the position vectors $X_E(t)$, $X_{obs}(t)$ and X_{E-obs}, in both cases (the fictitious Earth and the true Earth) is because the moving coordinate system (o, x_1, x_2), whose axis x_1 is always in the same direction of the vector $X_E(t)$, is rotating with an angular velocity (either Ω_{circ} or Ω_{Ell}), which is the same as the angular velocity of the Earth's trajectory (either circular or elliptical).

The first part of the Equation of Time is calculated by using the following equation:

$$\Delta t_{\mathrm{first}} = \frac{\Omega_{\mathrm{Circ}}\,t - \theta(t)}{\omega_{\mathrm{Circ}}} \equiv [\text{ minutes }] \quad (24)$$

In Fig. 9, the left panel shows the first component of the Equation of Time (correction due to the eccentricity of the Earth's orbit). The angle θ(t) in Eq. (24) is obtained by using the polar angle θ(t), which is the independent variable, and the PSA algorithm (for the year 2013), in which instead of using the polar angle θ(t), the ecliptic longitude $\hat{\theta}$ is used. In the PSA algorithm, instead of using the angle spanned by the fictitious Earth, Ω_{Circ} t, the mean longitude L is used, while in the analytical approach, the time t in the expression $\Omega_{Circ\ t}$, is obtained by the analytical solution. In the left panel of Fig. 9, it is observed that when the Earth is close to the Sun (January, February), the first component of the Equation of time is negative, i.e. the angular velocity of the true Earth ($\Omega_{Ell}(t) = d\theta(t)/dt$) is higher than the constant angular velocity of the standard fictitious Earth (Ω_{Circ}), as it was shown in the results of our imaginary scenery. However, when the Earth is far from the Sun (July, August) the angular velocity of the true Earth is lower than that of the fictitious Earth, then the first correction to the Equation of time is positive.

3.3 Second component of the equation of time: correction due to the Earth's rotation axis tilt

The correction of time due to the tilt of the Earth's rotation axis relative to the ecliptic plane, is based on the variation, with respect to the time of the declination angle β(t). In order to understand the contribution of the tilt angle on the equation of time, we propose a model based on two Earths: (1) the

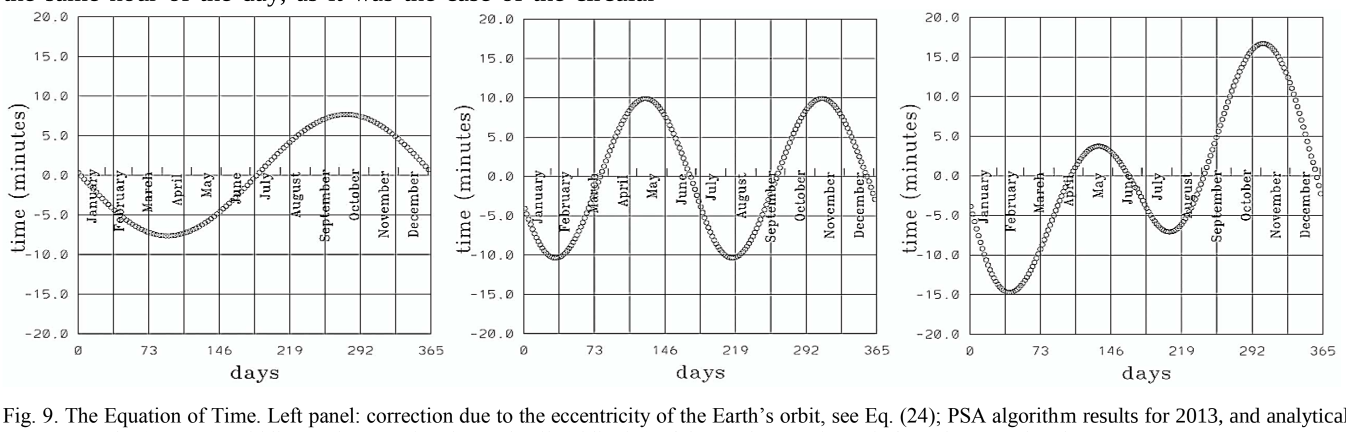

Fig. 9. The Equation of Time. Left panel: correction due to the eccentricity of the Earth's orbit, see Eq. (24); PSA algorithm results for 2013, and analytical results (see text). Middle panel: correction due to the tilt (obliquity) of the Earth's rotation axis from the normal to the ecliptic plane. Right panel: the Equation of Time as the sum of the two corrections shown in the left and middle panels, see Eq. (35).

standard fictitious Earth, and (2) the true Earth. In this model, it is assumed that as the Earth rotates about its rotation axis, Sun draws a curved line on the surface of each Earth. On one hand, in the fictitious Earth, it is assumed that, for a certain initial condition (noon of one selected day, that is when $\rho = 0$ radians), the latitude angle $\hat{\beta}$ (which is the value of the declination angle $\beta(t)$ at noon of the selected day) remains constant as the Earth rotates about its rotation axis, from noon of the selected day to noon of the next day (that is from $\rho = 0$ radians to $\rho = 2\pi$ radians). Along this fictitious day, it is apparent that the Sun generates on the surface of the Earth, a closed circular path that is parallel to the Earth's equatorial plane. The radius $\hat{r}$ (that is normal to the fictitious Earth's rotation axis) of this closed circular path is given as:

$$\hat{r} = r_f \cos\hat{\beta}, \tag{25}$$

where, r_f is the unit radius of the fictitious Earth. Then, after 24 hours, from noon of the selected day, and maintaining constant the latitude angle $\hat{\beta}$, the Sun has traveled on the surface of the fictitious Earth a total circular distance $\hat{d}_{f\ ict}$ given by

$$\hat{d}_{\text{fict}} = 2\pi\hat{r} = 2\pi r_f \cos\hat{\beta} \tag{26}$$

On the other hand, the closed curved path of the Sun on the surface of the true Earth is governed, not only by the longitude angle $\rho(t)$, but also by the declination angle $\beta(t)$. For the true Earth model, let us assume again as initial condition, that at noon (that is when $\rho = 0$ radians) of the day that was selected for the fictitious Earth, the Sun is at the declination angle $\beta(t)$, that is equal to the latitude angle $\hat{\beta}$ of the fictitious Earth (which remains constant until the next noon). Then, between two times $t_1$and t_2 (where $\Delta t = t_2 - t_1$=1 minute, corresponds to the time increment that the Sun's position (on the surface of the true Earth), has changed, not only due to the rotation of the Earth (given by the angle $\rho(t)$), but also due to the variation of the declination angle $\beta(t)$. To calculate the Sun's curved path on the surface of the true Earth between the two points P_1 (at t_1) and P_2 (at t_2), the spanned angle σ_{12} (the central angle) between these two points is evaluated either as:

$$\sigma_{12} = \cos^{-1}\left(\sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos(\rho_2 - \rho_1)\right), \tag{27}$$

or as:

$$\sigma_{12} = 2\sin^{-1}\left(\frac{C}{2}\right) \tag{28}$$

where, the variable C is given as:

$$C = \sqrt{(\Delta X)^2 + (\Delta Y)^2 + (\Delta Z)^2} \tag{29}$$

and

$$\Delta X = \cos\beta_2 \cos\rho_2 - \cos\beta_1 \cos\rho_1,$$
$$\Delta Y = \cos\beta_2 \sin\rho_2 - \cos\beta_1 \sin\rho_1, \Delta Z = \sin\beta_2 - \sin\beta_1 \tag{30}$$

The sub-indexes 1 and 2 mean that the angles $\beta(t)$ and $\rho(t)$ are evaluated at the times t_1 and t_2, respectively. The longitude angle spanned by the Earth's rotation after 1 minute is given by $\Delta\rho = \rho_2 - \rho_1$= 0.00436 radians ≡ 0.25 degrees.

The distance d^{12} (arc length) traveled by the Sun on the surface of the true Earth, between the two points P_1 and P_2, is obtained as:

$$d^{12} = r_t \sigma_{12}, \tag{31}$$

where, r_t is the unit radius of the true Earth. Then, after 24 hours (or after r=2π radians), the Sun has generated on the surface of the true Earth an helical (spiral) closed curved path whose total length $\hat{d}_{\text{true}}$ is given by the sum of each of the small distances d^{12}, traveled by the Sun, between each pair of points P_1 and P_2. As Δt = 1 minute, one day is constituted by 1440 time increments, hence along one day we have np=1440 pair of points P_1 and P_2. Hence, the total helical (spiral) distance traveled by the Sun on the surface of the true Earth (along 1440 minutes), is given as:

$$\hat{d}_{\text{true}} = \sum_{i=1}^{np} d_i^{12} \tag{32}$$

In the previous analysis, it has been considered that the radii r_f (see Eq. 26) and r_t (see Eq. 31) of both Earths is unitary, then both of the distances $\hat{d}_{\text{fict}}$ and $\hat{d}_{\text{true}}$ are expressed in radians, hence the difference $\hat{d}_{\text{diff}} = \hat{d}_{\text{fict}} - \hat{d}_{\text{true}}$ must be converted to minutes by using the following relationship:

$$\hat{d}_{\text{diff}} = \left(\hat{d}_{\text{fict}} - \hat{d}_{\text{true}}\right)[\text{ radians }]$$
$$= \left(\frac{24}{2\pi}\right)(60)\left(\hat{d}_{\text{fict}} - \hat{d}_{\text{true}}\right) \equiv [\text{ minutes }] \tag{33}$$

Consequently, the time correction due to the Earth's rotation axis tilt for one day that begins at $\rho(t)$=0 radian, and at $\hat{\beta} = \beta(t)$ (for the fictitious Earth) and at $\beta(t)$ (for the true Earth), and finishes (after 24 hours or 1440 minutes) at $\rho(t) = 2\pi$ radians and at $\hat{\beta}$ (for the fictitious Earth) and at the corresponding declination angle $\beta(t)$ for the true Earth, is given by the following equation:

$$\Delta t_{\text{second}} = \pi^2 \hat{d}_{\text{diff}} \equiv [\text{ minutes }] \tag{34}$$

where, the scaling factor π^2 is introduced to reproduce the second component of the Equation of Time published in the literature. Middle panel of Fig. 9 shows the second component of the Equation of time (see Eq. 34) as a function of time (days of the year). It is observed that the second correction of time is zero for four days: the two equinoxes and the two solstices. In these particular days, the Sun generates on the surface of the true Earth, circular paths (not spirals) similar to the circular paths generated by the Sun on the surface of the fictitious Earth. In the middle panel of Fig. 9, it is also observed that when the Sun travels from south to north, from the March equinox, to the June solstice (that is from March 20 to June 21), the daily spiral distance traveled by the Sun, $\hat{d}_{\text{true}}$ on the surface of the true Earth, is smaller than the daily distance, $\hat{d}_{\text{fict}}$ traveled by the Sun on the surface of the fictitious Earth, then the Sun on the true Earth, is ahead from the Sun on the fictitious Earth, hence the second correction of time Δt_{second} is positive. The same situation appears when the Sun travels from north to south, from the September equinox to the December solstice (from Sept. 2019 to Dec.2021).

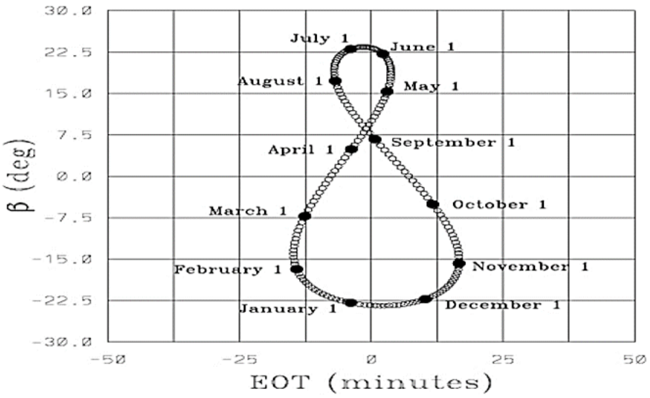

Fig. 10 Analemma for the year 2013. The Equation of Time, see Eq. (35) (as the abscissa) and the declination angle β(t), as the ordinate.

However, when the Sun is traveling from north to south, from the June solstice to the September equinox (that is from June 21 to September 19), the daily spiral distance traveled by the Sun, $\tilde{d}_{\text{true}}$, on the surface of the true Earth, is greater than the daily distance $\hat{d}_{\text{fict}}$, traveled by the Sun on the surface of the fictitious Earth, then the Sun on the true Earth, is delayed with respect the Sun on the fictitious Earth, hence the second correction of time is negative. The same situation appears when the Sun travels from south to north, from the December solstice to the March equinox (from December 21 to March 20). The sum of the times Δt_{first} , see Eq. (24) and Δt_{second}, see Eq. (34), is the so called Equation (correction) of Time given by:

$$EOT = \Delta t_{\text{first}} + \Delta t_{\text{second}} \equiv [\text{ minutes }] \qquad (35)$$

This Equation of Time (EOT) is plotted in Fig. 9 (right most panel) as function of time (days along the year).

4. Solar Analemma

An analemma is a diagram showing the position of the Sun in the sky as seen from a fixed location on Earth at the same mean solar time. As the solar position varies over the course of a year, a line joining the solar position, for the same date and time of every month of the year resembles a figure like 8 (eight) [2,3]. Figure 10 shows the graph of the solution of the Equation of Time, see Eq. (35) (as the abscissa) and the declination angle, β(t) (as the ordinate) at noon (ρ = 0 radians) of each day along the year 2013 to yield the corresponding analemma. Note that the analemma of Fig. 10, does not depend on the latitude at which an observer is located on the Earth's surface. Fig. 11, shows the analemma calculated for an observer located at the latitude δ=40.73º for the year 2013. Left panel shows the analemma at 10:00 A.M., middle panel shows the analemma at noon, and the right panel shows the analemma at 4:00 P.M. It is important to point out that the solar azimuth angle shown as the abscissa in the panels of Fig. 11, is the modified solar azimuth angle $\hat{\alpha}_{\text{a}}$. This angle is obtained by correcting Eq. (14) with the Equation of Time (EOT) (see Eq. 35). That is, the abscissa of the analemma of Fig. 11, is obtained by using the following relationship:

$$\hat{\alpha}_{\text{a}} = \alpha_{\text{a}} + (0.25)\ \text{EOT} \qquad (36)$$

where, the coefficient 0.25 is the conversion factor from minutes to degrees.

5. Results and Discussion

In an earlier companion paper diverse computational methodologies were presented to calculate the trajectory of the Sun in the sky of an observer located on the Earth's surface. In this paper, the location of the North Star has been successfully calculated in a Cartesian coordinate system, which is a familiar coordinate system for the engineers.

Additionally, the position vector from the Earth to the Sun, γ(t) and the declination angle, β(t) as functions of time, were obtained by using an engineering approach. Standard transformations of the involved vectors (the position vector of the observer and the Sun's position vector) have been obtained by performing simple rotations of the Cartesian coordinate system. The horizon plane of the observer, the solar zenith angle α_z, the solar elevation angle α_e, and the solar azimuth angle measured from north of the observer's horizon plane α_a have been obtained by using the proposed simple engineering approach. It has been pointed out that when the Sun is north of the observer, a different interpretation of the

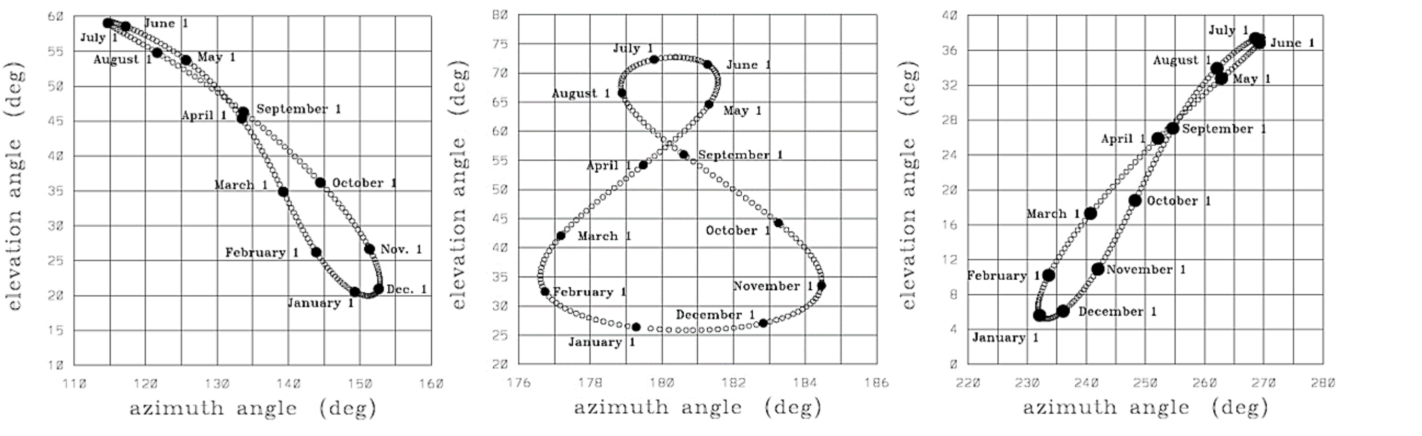

Fig. 11 Analemma for the year 2013. The observer is located at a latitude δ=40.73º. The solar azimuth angle measured from north α_a (degrees), see Eq. (14), as the abscissa and the solar elevation angle α_e (degrees), see Eq. (7), as the ordinate. Left panel: Analemma at 10:00 A.M. Middle panel: Analemma at noon. Right panel: Analemma at 4:00 P.M.

solar azimuth angle is needed, that is, when the declination angle $\beta(t)$ is greater than the latitude of the observer δ, the solar azimuth angle $\alpha_a(t)$ must be defined within the interval $-90^o < \alpha_a(t) < 90^o$

The two components of the Equation of Time have been explained in detail (from the physical point of view), and a new relationship to obtain the correction of time due to the Earth's rotation axis tilt has been presented.

For a graphic representation of the Equation of Time, see Fig. 9, and for a plot of the solar declination angle as a function of time, see Fig. 10.

The analemma for three distinct hours of the day (of the year 2013), for an observer located at the Earth's latitude $\delta = 40.73^o$ have been shown (see Fig. 11, corresponding to the New York city). The methodology presented here can be easily applied to other latitude locations and other calendar years.

The information included in this paper should be considered as an important source of reference for the solar energy engineers, who need to accurately know the position of the Sun throughout the year. The equations presented in this paper can be easily used by engineers instead of the sophisticated mathematical equations used by astronomers and astrophysicist.

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