The Nucleus **52**, *No. 3 (2015) 115-122*

www.thenucleuspak.org.pk

The Nucleus

ISSN 0029-5698 (Print)

ISSN 2306-6539 (Online)

On Bipolar Valued Fuzzy k-Ideals in Hemirings

T. Mahmood* and A. Ejaz

Department of Mathematics & Statistics, International Islamic University Islamabad, Pakistan

tahirbakhat@yahoo.com; asim_ejaz2005@yahoo.com

ARTICLE INFO

Article history:

Received : 7 January, 2015

Revised : 6 August, 2015

Accepted : 7 August, 2015

Keywords:

bipolar valued fuzzy sets,
k-intrinsic product,
bipolar valued fuzzy k-ideals

ABSTRACT

In this paper we discuss some results associated with Bipolar valued fuzzy k -ideals of hemirings. We also define bipolar valued fuzzy k-intrinsic product and characterize k-hemiregularhemirings by using their bipolar valued fuzzyk -ideals.

1. Introduction

Theory of semirings, as a generalization of associative rings and of distributive lattice, was introduced by Vandiver [1] in 1934. The structure was found very useful in information sciences and theoretical physics [2, 3]. The semirings with commutative "+" and zero element are called hemirings. Ideals of hemirings and semirings, as in rings, play very crucial role. But the ideals of hemirings and semirings have no analogue results to ring theory in general. Henriksen [4] addressed the problem and introduced k-ideals. These ideals have the property that a k-ideals in a semiring R, turns to ring ideal when R turns to ring.

In 1965, Zadeh [5] popularized the concept of fuzzy sets. Since then the concept of fuzzy set has been extensively used in many branches of Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [6] and he introduced the notion of fuzzy subgroups. In [7], J. Ahsan investigated the concept of fuzzy semirings (See also [8]). The fuzzy algebraic structures play a vital role in Mathematics with wide applications in many other branches such as computer sciences, theoretical physics, information sciences, control engineering, topological spaces and coding theory [9].

For a fuzzy set, its degree of membership expresses its degree of containment of elements to it. Sometimes, the degree of membership also means the degree of satisfaction of elements to some property or constraint corresponding to a fuzzy set [10]. Keeping in view this notion, the membership degree 0 is in general assigned to those elements of the set which do not satisfy some property. In the usual study of fuzzy set representation the elements with membership degree 0 are usually regarded as having the same characteristic. However it is observed that sometimes such type of thinking does not work. For example consider a fuzzy set *" young "*, defined on age domain [0, 100]. However it seems to be possible that the ages 50 and 90 both could have membership degree 0, but both of them have different contrary characteristics, for example we may say that age 90 is more apart from the property *"young"* rather than age 50 (see [11]). In such cases the usual fuzzy set does not help to differentiate between irrelevant elements and contrary elements. Keeping in view these facts Lee, proposed an extension of fuzzy sets named bipolar valued fuzzy sets (BVFS). In [12] Jun and song applied the notion of BVFS to BCH-algebras and in [13] Jun and Park discussed filters of BCH-algebras based on BVFS. In [14] K. J. Lee, used the notion of BVFS and worked on bipolar valued fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. More recently, in [15] BVFS are discussed in subgroups. In this paper we discuss some results associated with Bipolar valued fuzzy k-ideals of hemirings. We also define BVF k-intrinsic product and characterize k-hemiregularhemirings by the properties of their BVF k-ideals.

2. Preliminaries

For the terms and notions which are not defined, we refer to [2, 7, 15]. Throughout R will mean a hemiring, $P(R)$ will mean set of all subsets of R,$F(R)$will mean set of all fuzzy subsets of $R, LI(R)$ will mean set of all left k-

* Corresponding author

ideals of R, $FLI(R)$ will mean set of all BVF left k-ideals of R, $RI(R)$ will mean set of all right k-ideals of R, $FRI(R)$ will mean set of all BVF right k-ideals of R and $FI(R)$ will mean set of all BVF k-ideals of R.

2.1 Definition

Let $A \subseteq R$, such that $A \neq \emptyset$. Then the set

$\overline{A} = \{x \in R : x + a_1 = a_2 \text{ for some } a_1, a_2 \in A\}$ is called k-closure of A.

2.2 Definition [16]

Let $\lambda, \mu \in F(R)$, then their k-intrinsic product is denoted and defined as

$$(\lambda \otimes_k \mu)(x) = \bigvee_{x+\Sigma_{i=1}^{m} a_i b_i = \Sigma_{j=1}^{n} a'_j b'_j} \left\{ \begin{array}{l} \left(\overset{m}{\underset{i=1}{\wedge}} \lambda(a_i) \right) \wedge \\ \left(\overset{m}{\underset{i=1}{\wedge}} \mu(b_i) \right) \wedge \\ \left(\overset{n}{\underset{j=1}{\wedge}} \lambda\left(a'_j\right) \right) \wedge \\ \left(\overset{n}{\underset{j=1}{\wedge}} \mu\left(b'_j\right) \right) \end{array} \right\}$$

and $(\lambda \otimes_k \mu)(x) = 0$ if it is not possible to express x as

$x + \Sigma_{i=1}^{m} a_i b_i = \Sigma_{j=1}^{n} a'_j b'_j$.

2.3 Definition [17]

If for a hemiring R is such that for each of its element r, we have $a, b \in R$ such that $r + rar = rbr$, then R is called k-hemiregular hemiring.

2.4 Example

Let $R = No \cup \{\infty\}$. Then for all $x, y \in R$, define the operations

$$x \oplus y = \max\{x, y\}$$

and

$$x \otimes y = \min\{x, y\}$$

Then $(R, \oplus, \otimes)$ is a k-hemiregular hemiring.

2.5 Lemma [17]

R is k-hemiregular if and only if (iff) for any $I \in RI(R)$ and $L \in LI(R)$ we have $\overline{IL} = I \cap L$.

3. Bipolar-Valued Fuzzy (BVF) Subsets in Hemirings

In this section we will introduce the notion of BVFS of R and investigate some basic properties.

3.1 Definition [11]

For any universe X BVFS B of X is an object of the form

$B = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in X\}$,

where $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1,0]$.

3.2 Definition

A BVF subset $B = (\mu^+, \mu^-)$ of a hemiring R is called BVF k-subhemiring of R if for all $p, q, r, s, t \in R$.

$(B1)$ $\mu^+(p+q) \geq \min\{\mu^+(p), \mu^+(q)\}$,

$(B2)$ $\mu^+(pq) \geq \min\{\mu^+(p), \mu^+(q)\}$,

$(B1')$ $\mu^-(p+q) \leq \max\{\mu^-(p), \mu^-(q)\}$,

$(B2')$ $\mu^-(pq) \leq \max\{\mu^-(p), \mu^-(q)\}$.

$(B3)$ $s + r = t \to \mu^+(s) \geq \min\{\mu^+(r), \mu^+(t)\}$

$(B3')$ $s + r = t \to \mu^-(s) \leq \max\{\mu^-(r), \mu^-(t)\}$

3.3 Definition

For a BVF subset $B = (\mu^+, \mu^-)$ of a hemiring R, $B \in FRI(R)$ (resp. $B \in FRI(R)$) if it satisfies $(B1), (B1'), (B3).(B3')$ and

$(B4)$ $\mu^+(pq) \geq \mu^+(p)$ (resp. $(B5)$ $\mu^+(pq) \geq \mu^+(q)$),

$(B4')$ $\mu^-(pq) \leq \mu^-(p)$ (resp. $(B5')$ $\mu^-(pq) \leq \mu^-(q)$).

for all $p, q \in R$.

3.4 Theorem

For a BVF subset $B = (\mu^+, \mu^-)$ of a hemiring R,

$B \in FI(R)$, Iff $B \in FLI(R)$ and $B \in FRI(R)$.

3.5 Remark

If B is BVF ideal of R, then it is not necessary that $B \in FI(R)$.

Example 3.6

Let $R = \{0, s, t\}$, with addition "+" and multiplication "·" are defined as follows:

+	0	S	T
0	0	S	T
S	S	0	T
T	T	T	0

·	0	s	t
0	0	0	0
S	0	0	0
T	0	0	t

Then R is a hemiring. Now let $B = (\mu^+, \mu^-)$ be a BVF of R, defined by

$$\mu^+(0) = \mu^+(t) = 0.75, \mu^+(s) = 0.55$$
$$\mu^-(0) = \mu^-(t) = -0.65, \mu^-(s) = -0.43$$

Then $B = (\mu^+, \mu^-)$ is BVF ideal of R, but B is not in $FI(R)$.

3.7 Definition

Let $B_1 = (\mu_1^+, \mu_1^-)$ and $B_2 = (\mu_2^+, \mu_2^-)$ be two BVF subsets of R. Then the symbols $B_1 \vee B_2$ and $B_1 \wedge B_2$ means the following BVF subsets of R.

$$B_1 \vee B_2 = (\max\{\mu_1^+, \mu_2^+\}, \min\{\mu_1^-, \mu_2^-\})$$
$$B_1 \wedge B_2 = (\min\{\mu_1^+, \mu_2^+\}, \max\{\mu_1^-, \mu_2^-\})$$

3.8 Proposition

Let $B_1 = (\mu_1^+, \mu_1^-)$, $B_2 = (\mu_2^+, \mu_2^-) \in FLI(R)$(resp. $FRI(R)$). Then $B_1 \wedge B_2 \in FLI(R)$(resp. $FRI(R)$).

3.9 Example

Let $R = \{0, 1, 2, 3, 4, 5\}$ be a hemiring of residue classes modulo 6. Now let $B_1 = (\mu_1^+, \mu_1^-), B_2 = (\mu_2^+, \mu_2^-)$ be a BVF of R, defined by

$$\mu_1^+(0) = \mu_1^+(2) = \mu_1^+(4) = 0.75,$$
$$\mu_1^+(1) = \mu_1^+(3) = \mu_1^+(5) = 0.5$$
$$\mu_1^-(0) = \mu_1^-(2) = \mu_1^-(4) = -0.9,$$
$$\mu_1^-(1) = \mu_1^-(3) = \mu_1^-(5) = -0.6$$

and

$$\mu_2^+(0) = \mu_2^+(3) = 0.95,$$
$$\mu_2^+(1) = \mu_2^+(2) = \mu_2^+(4) = \mu_2^+(5) = 0.25$$
$$\mu_2^-(0) = \mu_2^-(3) = -0.87,$$
$$\mu_2^-(1) = \mu_2^-(2) = \mu_2^-(4) = \mu_2^-(5) = -0.34$$

Then $B_1, B_2 \in FI(R)$ but $B_1 \vee B_2 \notin FI(R)$.

3.10 Definition [3]

For any complex A in X, the bipolar valued characteristic function is denoted and given by

$C_A = (C_A^+, C_A^-)$, where

$$C_A^+(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$C_A^-(x) = \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

3.11 Proposition

For a subset A of R, $A \in LI(R)$(resp. $RI(R)$) iff $C_A \in FLI(R)$(($resp. RI(R)$)).

3.12 Theorem

Every BVF k-ideal of R is a BVF k-subhemiring of R.

3.13 Definition

Let $B_1 = (\mu_1^+, \mu_1^-)$ and $B_2 = (\mu_2^+, \mu_2^-)$ be two BVFS of R. Then

$$(B_1 \otimes_k B_2)(x) = ((\mu_1^+ \otimes_k \mu_2^+)(x), (\mu_1^- \otimes_k \mu_2^-)(x))$$

Where

$$(\mu_1^+ \otimes_k \mu_2^+)(x) = \bigvee_{x+\Sigma_{i=1}^m a_i b_i = \Sigma_{j=1}^n a_j' b_j'} \left\{ \begin{array}{l} \left(\bigwedge_{i=1}^m \mu_1^+(a_i)\right) \wedge \\ \left(\bigwedge_{i=1}^m \mu_2^+(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^n \mu_1^+(a_j')\right) \wedge \\ \left(\bigwedge_{j=1}^n \mu_2^+(b_j')\right) \end{array} \right\}$$

and $(\mu_1^+ \otimes_k \mu_2^+)(x) = 0$ if it is not possible to express x as $x + \Sigma_{i=1}^m a_i b_i = \Sigma_{j=1}^n a_j' b_j'$

$$(\mu_1^- \otimes_k \mu_2^-)(x) = \bigwedge_{x+\Sigma_{i=1}^m a_i b_i = \Sigma_{j=1}^n a_j' b_j'} \left\{ \begin{array}{l} \left(\bigvee_{i=1}^m \mu_1^-(a_i)\right) \vee \\ \left(\bigvee_{i=1}^m \mu_2^-(b_i)\right) \vee \\ \left(\bigvee_{j=1}^n \mu_1^-(a_j')\right) \vee \\ \left(\bigvee_{j=1}^n \mu_2^-(b_j')\right) \end{array} \right\}$$

and $\left(\mu_1^- \otimes_k \mu_2^-\right)(x) = 0$ if it is not possible to express x as $x + \Sigma_{i=1}^m a_i b_i = \Sigma_{j=1}^n a_j' b_j'$

3.14 Definition

Let $B_1 = (\mu_1^+, \mu_1^-)$ and $B_2 = (\mu_2^+, \mu_2^-)$ be two BVFS of R. Then $B_1 \subseteq B_2$ iff $\mu_1^+ \subseteq \mu_2^+$ and $\mu_1^- \supseteq \mu_2^-$.

3.15 Lemma

If $B_1 = (\mu_1^+, \mu_1^-) \in FRI(R)$) and $B_2 = (\mu_2^+, \mu_2^-) \in FLI(R)$). Then $B_1 \otimes_k B_2 \subseteq B_1 \cap B_2$.

Proof : Let $x \in R$ if it is not possible to express x as

$x + \Sigma_{i=1}^n a_i b_i = \Sigma_{j=1}^m c_j d_j$ for any $a_i, b_i, c_j, d_j \in R,$ then

$$(B_1 \otimes_k B_2)(x) = 0 \le B_1(x) \wedge B_2(x) = (B_1 \cap B_2)(x)$$

Otherwise,

$$\mu_1^+(x) \ge \mu_1^+\left(\Sigma_{i=1}^n a_i b_i\right) \wedge \mu_1^+\left(\Sigma_{j=1}^m c_j d_j\right)$$
$$\mu_1^-(x) \le \mu_1^-\left(\Sigma_{i=1}^n a_i b_i\right) \vee \mu_1^-\left(\Sigma_{j=1}^m c_j d_j\right)$$

and

$$\mu_2^+(x) \ge \mu_2^+\left(\Sigma_{i=1}^n a_i b_i\right) \wedge \mu_2^+\left(\Sigma_{j=1}^m c_j d_j\right)$$
$$\mu_2^-(x) \le \mu_2^-\left(\Sigma_{i=1}^n a_i b_i\right) \vee \mu_2^-\left(\Sigma_{j=1}^m c_j d_j\right)$$

Now

$$\begin{aligned}\left(\mu_1^+ \wedge \mu_2^+\right)(x) &= \mu_1^+(x) \wedge \mu_2^+(x) \\ &\ge \left\{\begin{matrix}\mu_1^+\left(\Sigma_{i=1}^n a_i b_i\right) \wedge \mu_1^+\left(\Sigma_{j=1}^m c_j d_j\right) \\ \wedge \mu_2^+\left(\Sigma_{i=1}^n a_i b_i\right) \wedge \mu_2^+\left(\Sigma_{j=1}^m c_j d_j\right)\end{matrix}\right\} \\ &\ge \wedge_{i,j}\left\{\begin{matrix}\mu_1^+(a_i b_i) \wedge \mu_1^+(c_j d_j) \\ \wedge \mu_2^+(a_i b_i) \wedge \mu_2^+(c_j d_j)\end{matrix}\right\} \\ &\ge \wedge_{i,j}\left\{\mu_1^+(a_i) \wedge \mu_1^+(c_j) \wedge \mu_2^+(bi) \wedge \mu_2^+(d_j)\right\}.\end{aligned}$$

and

$$\begin{aligned}\left(\mu_1^- \vee \mu_2^-\right)(x) &= \mu_1^-(x) \vee \mu_2^-(x) \\ &\le \left\{\begin{matrix}\mu_1^-\left(\Sigma_{i=1}^n a_i b_i\right) \vee \mu_1^-\left(\Sigma_{j=1}^m c_j d_j\right) \vee \\ \mu_2^-\left(\Sigma_{i=1}^n a_i b_i\right) \vee \mu_2^-\left(\Sigma_{j=1}^m c_j d_j\right)\end{matrix}\right\} \\ &\le \vee_{i,j}\left\{\begin{matrix}\mu_1^-(a_i b_i) \vee \mu_1^-(c_j d_j) \vee \\ \mu_2^-(a_i b_i) \vee \mu_2^-(c_j d_j)\end{matrix}\right\} \\ &\le \vee_{i,j}\left\{\begin{matrix}\mu_1^-(a_i) \vee \mu_1^-(c_j) \vee \\ \mu_2^-(bi) \vee \mu_2^-(d_j)\end{matrix}\right\}.\end{aligned}$$

Since above expression holds for any $a_i, b_i, c_j, d_j \in R$ and for all i, j therefore

$$\left(\mu_1^+ \wedge \mu_2^+\right)(x) \ge \bigvee_{x+\Sigma_{i=1}^n a_i b_i = \Sigma_{j=1}^m c_j d_j} \left\{\begin{matrix}\wedge_{ij}[\mu_1^+(a_i) \wedge \mu_1^+(c_j) \\ \wedge \mu_2^+(b_i) \wedge \mu_2^+(d_j)]\end{matrix}\right\}$$
$$= \left(\mu_1^+ \otimes_k \mu_2^+\right)(x)$$
$$\left(\mu_1^- \vee \mu_2^-\right)(x) \le \bigwedge_{x+\Sigma_{i=1}^n a_i bi = \Sigma_{j=1}^m c_j d_j} \left\{\begin{matrix}\vee_{ij}[\lambda^-(a_i) \vee \lambda^-(c_j) \\ \vee \mu^-(b_i) \vee \mu^-(d_j)]\end{matrix}\right\}$$
$$= \left(\mu_1^- \otimes_k \mu_2^-\right)(x)$$

$$\Rightarrow B_1 \otimes_k B_2 \subseteq B_1 \cap B_2 .$$

3.16 Lemma

Let B_1, B_2, B_3, B_4 be any BVFS of R such that $B_1 \subseteq B_2$. and $B_2 \subseteq B_4$. Then $B_1 \otimes_k B_2 \subseteq B_3 \otimes_k B_4$.

Proof : If x cannot be written in the form

$x + \sum_{i=1}^n a_i b_i = \sum_{i=1}^m c_j d_j$ for any $a_i, b_i, c_j, d_j \in R$ then

$$(B_1 \otimes_k B_2)(x) = 0 = (B_3 \otimes_k B_4)(x).$$

Otherwise since $B_1 \subseteq B_3$ and $B_2 \subseteq B_4$ so

$$\mu_1^- \supseteq \mu_3^- \quad \mu_1^+ \subseteq \mu_3^+$$
$$\mu_2^- \supseteq \mu_4^- \quad \mu_2^+ \subseteq \mu_4^+$$

and hence for all $x \in R$

$$\left(\mu_1^+ \otimes_k \mu_2^+\right)(x) = \bigvee_{x+\Sigma_{i=1}^n a_i b_i = \Sigma_{j=1}^m c_j d_j} \left\{\begin{matrix}\wedge_{i,j}[\mu_1^+(a_i) \wedge \\ \mu_1^+(c_j) \wedge \mu_2^+(b_i) \\ \wedge \mu_2^+(d_j)]\end{matrix}\right\}$$

$$\leq \bigvee_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}c_jd_j} \left\{ \begin{matrix} \wedge_{i,j}[\mu_3^+\left(a_i\right)\wedge \\ \mu_3^+\left(c_j\right)\wedge \mu_4^+\left(b_i\right) \\ \wedge \mu_4^+\left(d_j\right)] \end{matrix} \right\}$$

$$=\left(\mu_3^+ \otimes_k \mu_4^+\right)\left(x\right)$$

and

$$\left(\mu_1^- \otimes_k \mu_2^-\right)\left(x\right)= \bigwedge_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}c_jd_j} \left\{ \begin{matrix} \vee_{i,j}[\mu_1^-\left(a_i\right)\vee \\ \mu_1^-\left(c_j\right)\vee \mu_2^-\left(b_i\right) \\ \vee \mu_2^-\left(d_j\right)] \end{matrix} \right\}$$

$$\geq \bigwedge_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}c_jd_j} \left\{ \begin{matrix} \vee_{i,j}[\mu_3^-\left(a_i\right)\vee \\ \mu_3^-\left(c_j\right)\vee \mu_4^-\left(b_i\right) \\ \vee \mu_4^-\left(d_j\right)] \end{matrix} \right\}$$

$$=\left(\mu_3^- \otimes_k \mu_4^-\right)\left(x\right)$$

$\Rightarrow B_1 \otimes_k B_2 \subseteq B_3 \otimes_k B_4$.

3.17 Lemma

For a BVF subset $B=(\mu^+,\mu^-)$ of $R, B\in FLI(R)$ (resp. *FRI(R))* if for all $p,q,r,s,t \in R,$ we have

(i) $\mu^+(p+q)\geq \mu^+(p)\wedge \mu^+(q)$ and

$\mu^-(p+q)\leq \mu^-(p)\vee \mu^-(q))$

(ii) $C_R \otimes_k B \subseteq B$ (resp. $B \otimes_k C_R \subseteq B$)

(iii) $s+r=t \Rightarrow \mu^+(s)\geq \mu^+(r)\wedge \mu^+(t)$ and

$\mu^-(s)\leq \mu^-(r)\vee \mu^-(t)$

Proof: Let $B=(\mu^+,\mu^-)\in FLI(R).$ Then by definition (i) and (iii) are true. Now let $x \in R$ if it is not possible to express x as $x+\sum_{i=1}^{n}a_ib_i = \sum_{j=1}^{m}a_j^{'}b_j^{'}$ for any

$a_i, a_j^{'}, b_i, b_j^{'} \in R,$

Otherwise

$$\left(C_R \otimes_k B\right)\left(x\right)=\begin{pmatrix} \left(C_R^+ \otimes_k \mu^+\right)\left(x\right), \\ \left(C_R^- \otimes_k \mu^-\right)\left(x\right) \end{pmatrix}$$

$$\left(C_R^+ \otimes_k \mu^+\right)\left(x\right)= \bigvee_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \left\{ \wedge_{i,j} \begin{bmatrix} \mu^+(b_i) \\ \wedge \mu^+(b_j^{'}) \end{bmatrix} \right\}$$

$$\leq \bigvee_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \left\{ \wedge_{i,j} \begin{bmatrix} \mu^+(a_ib_i) \\ \wedge \mu^+(a_j^{'}b_j^{'}) \end{bmatrix} \right\}$$

$$\leq \bigvee_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \begin{bmatrix} \mu^+(\sum_{i=1}^{n}a_ib_i) \\ \wedge \mu^+(\sum_{j=1}^{m}a_j^{'}b_j^{'}) \end{bmatrix}$$

$$\leq \vee\left(\mu^+\left(x\right)\right)$$

$$=\mu^+\left(x\right).$$

and

$$\left(C_R^- \otimes_k \mu^-\right)\left(x\right)= \bigwedge_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \left\{ \vee_{i,j} \begin{bmatrix} \mu^-(b_i) \\ \vee \mu^-(b_j^{'}) \end{bmatrix} \right\}$$

$$\geq \bigwedge_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \left\{ \vee_{i,j} \begin{bmatrix} \mu^-(a_ib_i) \\ \vee \mu^-(a_j^{'}b_j^{'}) \end{bmatrix} \right\}$$

$$\geq \bigwedge_{x+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \begin{bmatrix} \mu^-(\sum_{i=1}^{n}a_ib_i) \\ \vee \mu^-(\sum_{j=1}^{m}a_j^{'}b_j^{'}) \end{bmatrix}$$

$$\geq \wedge\left(\mu^-\left(x\right)\right)$$

$$=\mu^-\left(x\right).$$

Hence $C_R \otimes_k B \subseteq B$.

Conversely, assume that $(i),(ii),(iii)$ hold for a BVF subset B of R, then to prove that $B \in FLI(R)$, we only have to show that $\mu^+(x_1x_2)\geq \mu^+(x_2)$ and $\mu^-(x_1x_2)\leq \mu^-(x_2)$ for all $x_1,x_2 \in R$. So let $x_1,x_2 \in R$. Then by (ii)

$$\mu^+\left(x_1x_2\right)\geq \left(C_R^+ \otimes_k \mu^+\right)\left(x_1x_2\right)$$

$$= \bigvee_{x_1x_2+\Sigma_{i=1}^{n}a_ib_i=\Sigma_{j=1}^{m}a_j^{'}b_j^{'}} \left\{ \wedge_{i,j}\left[\mu^+(b_i)\wedge \mu^+(b_j^{'})\right] \right\}$$

$\geq \left[\mu^{+}(x_2) \wedge \mu^{+}(x_2)\right]$ because $x_1x_2 + 0x_2 = x_1x_2$

$= \mu^{+}(x_2).$

and

$\mu^{-}(x_1x_2) \leq \left(C_R^{+} \otimes_k \mu^{-}\right)(x_1x_2)$

$= \bigwedge_{x_1x_2+\Sigma_{i=1}^{n} a_i b_i = \Sigma_{j=1}^{m} a_j' b_j'} \left\{\vee_{i,j}\left[\mu^{-}(b_i) \vee \mu^{-}(b_j')\right]\right\}$

$\leq \left[\mu^{-}(x_2) \vee \mu^{-}(x_2)\right]$

because $x_1x_2 + 0x_2 = x_1x_2$

$= \mu^{-}(x_2).$

Therefore $B \in FI(R)$.

3.18 Theorem

A hemiring R is k-hemiregular if and only if for any bipolar valued fuzzy right k-ideal A and bipolar valued fuzzy left k–ideal B of R, we have

$$\overline{AB} = A \cap B.$$

3.19 Lemma

Let $A, B \subseteq R.$ Then $C_A \otimes_k C_B = C_{\overline{AB}}$

Proof: Let $x \in R$.If $x \in \overline{AB}$, then $C^{+}_{\overline{AB}}(x) = 1,$

$C^{-}_{\overline{AB}} = -1$

and for $x + \sum_{i=1}^{n} a_i b_i = \sum_{j=1}^{m} a_j' b_j'$

$$C_A^{+}(a_i) = C_A^{+}(a_j') = C_B^{+}(b_i) = C_B^{+}(b_j') = 1$$
$$C_A^{-}(a_i) = C_A^{-}(a_j') = C_B^{-}(b_i) = C_B^{-}(b_j') = -1$$

and hence

$$(C_A \otimes_k C_B)(x) = \begin{pmatrix} \left(C_A^{+} \otimes_k C_B^{+}\right)(x), \\ \left(C_A^{-} \otimes_k C_B^{-}\right)(x) \end{pmatrix}$$

$$\left(C_A^{+} \otimes_k C_B^{+}\right)(x) = \bigvee_{x+\Sigma_{i=1}^{n} a_i b_i = \Sigma_{j=1}^{m} a_j' b_j'} \begin{Bmatrix} \wedge_{i,j}[C_A^{+}(a_i) \\ \wedge C_A^{+}(a_j') \wedge \\ C_B^{+}(b_i) \\ \wedge C_B^{+}(b_j')] \end{Bmatrix} = 1$$

and

$$\left(C_A^{-} \otimes_k C_B^{-}\right)(x) = \bigwedge_{x+\Sigma_{i=1}^{n} a_i b_i = \Sigma_{j=1}^{m} a_j' b_j'} \begin{Bmatrix} \vee_{i,j}[C_A^{+}(a_i) \\ \vee C_A^{+}(a_j') \vee \\ C_B^{+}(b_i) \vee \\ C_B^{+}(b_j')] \end{Bmatrix} = -1$$

Therefore whenever $x \in \overline{AB}$ then

$$\left(C_A^{+} \otimes_k C_B^{+}\right)(x) = \left(C^{+}_{\overline{AB}}\right)(x) = 1$$
$$\left(C_A^{-} \otimes_k C_B^{-}\right)(x) = \left(C^{-}_{\overline{AB}}\right)(x) = -1$$

and if $x \notin \overline{AB}$ then

$C_{\overline{AB}}(x) = 0$.

If possible, let $(C_A \otimes_k C_B)(x) \neq 0$ then

$$\bigvee_{x+\Sigma_{i=1}^{n} a_i b_i = \Sigma_{j=1}^{m} a_j' b_j'} \begin{Bmatrix} \wedge_{i,j}[C_A^{+}(a_i) \\ \wedge C_A^{+}(a_j') \\ \wedge C_B^{+}(b_i) \\ \wedge C_B^{+}(b_j')] \end{Bmatrix} \neq 0$$

$$\bigwedge_{x+\Sigma_{i=1}^{n} a_i b_i = \Sigma_{j=1}^{m} a_j' b_j'} \begin{Bmatrix} \vee_{i,j}[C_A^{-}(a_i) \\ \vee C_A^{-}(a_j') \vee \\ C_B^{-}(b_i) \vee \\ C_B^{-}(b_j')] \end{Bmatrix} \neq 0$$

Therefore there exist $p_i, q_i, p_j', q_j' \in R$ such that

$$x + \sum_{i=1}^{n} a_i b_i = \sum_{j=1}^{m} a_j' b_j'$$

and

$$\wedge_{i,j}\left[C_A^{+}(p_i) \wedge C_A^{+}(p'_i) \wedge C_B^{+}(q_i) \wedge C_A^{+}(q'_i)\right] \neq 0$$
$$\vee_{i,j}\left[C_A^{-}(p_i) \vee C_A^{-}(p'_i) \vee C_B^{-}(q_i) \vee C_A^{-}(q'_i)\right] \neq 0$$

Then obviously for all i and j

$$C_A^{-}(p_i) = C_A^{-}(p_j') = C_B^{-}(q_i) = C_B^{-}(q_j') = -1$$

and

$$C_A^{+}(p_i) = C_A^{+}(p_j') = C_B^{+}(q_i) = C_B^{+}(q_j') = 1$$

So for all i, j

$$C_A^{+}(p_i) = C_B^{+}(q_i) = 1, C_A^{-}(p_i) = C_B^{-}(q_i) = -1$$

and

$$C_A^+(p_j^{'}) = C_B^+(q_j^{'}) = 1, C_A^-(p_j^{'}) = C_B^-(q_j^{'}) = -1$$

$p_i \in A, q_i \in B\ \forall i$

and $p_j^{'} \in A, q_j^{'} \in B\ \forall j$

$\Rightarrow x \in \overline{AB}$. This contradicts $C_{\overline{AB}}(x) = 0$

Therefore whenever $x \notin \overline{AB}$ then again we have

$$(C_A \otimes_k C_B)(x) = 0 = C_{\overline{AB}}(x)$$

Hence it is proved that $C_A \otimes_k C_B = C_{\overline{AB}}$.

3.20 Theorem

A hemiring R is k-hemiregulariff for any

$B_1 \in FRI(R)$ and $B_2 \in FLI(R)$, we have

$$B_1 \otimes_k B_2 = B_1 \cap B_2$$

Proof : Let R be a k-hemi regular hemiring. Then by Lemma 3.15, $B_1 \otimes_k B_2 \subseteq B_1 \cap B_2$

For reverse containment, since R is k-hemiregular so for all $a \in R$, there exist $x_1, x_2 \in R$ such that $a + ax_1a = ax_2a.$

Now

$$(B_1 \otimes_k B_2)(a) = \begin{pmatrix} (\mu_1^+ \otimes_k \mu_2^+)(a), \\ (\mu_1^- \otimes_k \mu_2^-)(a) \end{pmatrix}$$

$$(\mu_1^+ \otimes_k \mu_2^+)(a) = \bigvee_{a+\Sigma_{i=1}^n a_i b_i = \Sigma_{j=1}^m c_j d_j} \left\{ \wedge_{ij} \begin{bmatrix} \mu_1^+(a_i) \\ \wedge \mu_1^+(c_j) \\ \wedge \mu_2^+(b_i) \\ \wedge \mu_2^+(d_j) \end{bmatrix} \right\}$$

$$\geq \begin{bmatrix} \mu_1^+(a) \wedge \mu_1^+(ax_1) \\ \wedge \mu_2^+(ax_2) \wedge \mu_2^+(a) \end{bmatrix}$$

$$\geq [\mu_1^+(a) \wedge \mu_1^+(a) \wedge \mu_2^+(a) \wedge \mu_2^+(a)$$

$$= [\mu_1^+(a) \wedge \mu_2^+(a)]$$

$$= (\mu_1^+ \cap \mu_2^+)(a).$$

and

$$(\mu_1^- \otimes_k \mu_2^-)(a) = \bigwedge_{a+\Sigma_{i=1}^n a_i b_i = \Sigma_{j=1}^m c_j d_j} \left\{ \vee_{ij} \begin{bmatrix} \mu_1^-(a_i) \\ \vee \mu_1^-(c_j) \\ \vee \mu_2^-(b_i) \\ \vee \mu_2^-(d_j) \end{bmatrix} \right\}$$

$$\leq \begin{bmatrix} \mu_1^-(a) \vee \mu_1^-(ax_1) \vee \mu_2^- \\ (ax_2) \vee \mu_2^-(a) \end{bmatrix}$$

$$\leq [\mu_1^-(a) \vee \mu_1^-(a) \vee \mu_2^-(a) \vee \mu_2^-(a)]$$

$$= [\mu_1^-(a) \vee \mu_2^-(a)] = (\mu_1^- \cup \mu_2^-)(a).$$

Thus

$$B_1 \cap B_2 \subseteq B_1 \otimes_k B_2.$$

Hence

$$B_1 \cap B_2 = B_1 \otimes_k B_2.$$

Conversely, let A and B be right and left k-ideals of R respectively, then their characteristic functions C_A and $C_B$are also BVF right and BVF left k-ideals of R respectively. Then by hypothesis

$$C_{\overline{AB}} = C_A \otimes C_B = C_A \cap C_B = C_{A \cap B}.$$

Thus $\overline{AB} = A \cap B$

Hence R is k-hemiregular

4. Conclusion

In this paper we discussed some results associated with Bipolar valued fuzzy k -ideals of hemirings. We also defined bipolar valued fuzzy k-intrinsic product and characterized k-hemiregular hemirings by using their bipolar valued fuzzy$k$–ideals. As future work Characterizations of semi-simplehemirings and Characterizations of intra-regular hemirings can be made by using different types of bi-polar valued fuzzy k-ideals.

References

[1] H.S. Vandiver, "Note on a simple type of algebra in which cancellation law of addition does not hold", Bull. Amer. Math. Soc., vol. 40, pp. 914-920, 1934.

[2] J.S. Golan, "Semirings and their applications", Kluwer Acad. Publ, 1999.

[3] U. Hebisch and H. J. Weinert, "Semirings: Algebraic Theory and Appl. in the Comp. Sci.", World Scientific, 1998.

[4] M. Henriksen, "Ideals in semirings with commutative addition", Amer. Math. Soc. Notices, 1958.

[5] L.A. Zadeh, "Fuzzy sets", Information and Control, vol. 8, pp. 338-353, 1965.

[6] A. Rosenfeld, "Fuzzy groups", J. Math. Anal. Appl., vol. 35, pp. 512-517, 1971.

[7] J. Ahsan, K. Saifullah and M. F. Khan, “Fuzzy semirings”, Fuzzy Sets and Systems, vol. 60, pp. 309-320, 1993.

[8] [8] J. Ahsan, “Semirings characterized by their fuzzy ideals”, J. Fuzzy Math., vol. 6, pp. 181-192, 1998.

[9] W. Wechler, “The concept of fuzziness in automata and language theory”, Akademicverlog, Berlin, 1978.

[10] H. J. Zimmermann, “Fuzzy Set Theory and its Applications”, Kluwer Nijhoff Publishing, 1985.

[11] K. M. Lee, “Bipolar-valued fuzzy sets and their operations”, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, proc., vol. 307312, 2000.

[12] Y. B. Jun and S.Z. Song, “Subalgebras and Closed Ideals of BCH-Algebras Based on Bipolar-Valued Fuzzy Sets”, Sci. Math. Jpn., vol. 37, pp. 427-437, 2008.

[13] Y. B. Jun and C. H. Park, “Filters of BCH-algebras based on bipolar-valued fuzzy sets”, Int. Mathematical Forum, vol. 4, pp. 631-643, 2009.

[14] K. J. Lee, “Bipolar Fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras”, Bulletin of Malaysian Mathematical Sciences Society, vol. 32, pp. 361-373, 2009.

[15] T. Mahmood and M. Munir, “On bipolar fuzzy subgroups”, World Applied Sciences Journal, vol. 27, pp. 1806-1811, 2013.

[16] M. Shabir, and R. Anjum, “Characterization of hemirings by their k-ideals”, Applied Mathematics, vol. 4, pp. 753-768, 2013.

[17] M. K. Sen and P. Mukhopadhyay, "Von NewumannRegularity in Semirings, "Kyungpook Mathematical Journal, vol. 35, pp. 249-258, 1995.