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ON DOMINATION NUMBER OF CARTESIAN PRODUCT OF EVEN CYCLES

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Let $\gamma(G)$ denote the domination number of the graph G and let $\gamma(G\Box H)$ denote the domination number of the Cartesian product of the graphs G and H. Here in this note; let C_3 denote the cycle with three vertices and similarly, let C_n denote the cycle with n vertices. The domination number of the Cartesian product of two even cycles C_m and C_n is characterized here, where $m<n$, with $m \geq 4$ such that

$$\gamma(C_m\Box C_n)=\frac{mn}{4}$$

if and only if 2 divides $\frac{mn}{4}$, that is, iff $2 \mid \frac{mn}{4}$.

Keywords: Cartesian product, Domination number, Vizing's conjecture

1. Introduction

A graph G is defined by a set of vertices $V(G)$ and an edge set $E(G)$ and an incidence relation which associates with each edge either one or two vertices called end vertices or end points [5]. A graph is simple if it has no loops and no multiple edges.

A set of vertices D of a graph G is called a *dominating set* if every vertex of G is dominated by some vertex in D. Equivalently, a set D of vertices of a graph G is dominating set if every vertex in $V(G)-D$ is adjacent to some vertex $V \in D$. The domination number of a graph G, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of a graph G. A dominating set D with $|D|=\gamma(G)$ is called the minimum dominating set [9].

The Cartesian product of simple graphs G and H is the graph $G\Box H$ whose vertex set is $V(G)\times V(H)$ and whose edge set is the set of all pairs $(a,x)(b,y) \in E(G\times H)$ whenever $x=y$ and $ab \in E(G)$ or $a=b$ and $xy \in E(H)$, that is

$$E(G\times H)=\left\{\{(a,x),(b,y)\} \mid \begin{array}{l} x=y \text{ and } ab \in G \\ a=b \text{ and } xy \in H \end{array}\right\}$$

For $x \in V(H)$, set $G_x=G\times\{x\}$ and for $a \in V(G)$, set $H_a=\{a\}\times H$, the sets G_x and H_a are called layers of G or H respectively [1,2]. For $n \geq 3$, the Cartesian product $C_n\Box K_2$ is polyhedral graph called the n-prism; the 3-prism, 4-prism, and 5-prism are commonly called the triangular prism, cube and the pentagonal prism .

In 2004, A. Kloboucar determined the total domination of the Cartesian product of paths, i.e., $P_5\Box P_n$ and $P_6\Box P_n$ such that $\gamma_t(P_5\Box P_n) = \left\lfloor \frac{3n+4}{2} \right\rfloor, n \neq 6$ and $\gamma_t(P_6\Box P_n)= \left\lfloor \frac{12n+21}{7} \right\rfloor$ [11]. Recently, in a private

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communication [10], Daniel Gonçalves, Alexandre Pinlou, Michaël Rao and Stéphan Thomassé calculated the domination number of all $n \times n$ grid graphs and proved the Chang's conjecture for every $16 \le n \le m$, $\gamma(G_{n,m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$ [10].

On domination theory of Cartesian product of graphs; there are two fundamental problems, one is the conjecture of Vizing, which is still open, stated in [1,2] such as $\gamma(G \square H) \ge \gamma(G)\gamma(H)$ that is the domination number of the Cartesian product of the two graphs is at least the product of their domination numbers and for many partial results see [3,4]. The other problem is to determine the domination number of certain Cartesian products of graphs [5,6]. Also this problem seems to be a difficult one and even for a subgraph of $P_m \square P_n$ is NP-complete and the problem itself is also open.

2. Main Results

Throughout this note, the vertices of the cycles are indexed as $0,1,2,...,n-1$. The Cartesian product grid generated by the product of two cycles is also indexed from the set $\{0,1,2,...,n-1\} \times \{0,1,2,...,n-1\}$.

Lemma 1. Let m and n be positive even integers with $m<n$ and $m \ge 4$, then there exists a minimum dominating set $D = \{I_0 \times J\} \cup \{I_1 \times K\} \cup \{I_2 \times L\} \cup \{I_3 \times P\} \cup ... \cup \{I_{m-1} \times J\} \cup ...$

Proof. Let D be the minimum dominating set of the Cartesian product of two even cycles C_m and C_n. As the Cartesian product contains m copies of C_n and conversely n copies of cycle C_m. Let $I = \{i \,|\, 0 \le i \le m-1\}$ be the set denoting i, the horizontal index which runs in the interval $0 \le i \le m-1$, hence $m-1$ of C_n-layers. Let each i represents a layer C_{n_i} with the total number of $m-1$ layers with each layer containing vertices of the dominating set D. Let $J = \{j \,|\, j \equiv 0 (\mathrm{mod}\, 4)\}$ and its Cartesian product with the set $I_0 = \{0\}$, that is, $I_0 \times J = \{(i_0, j) | i_0 \in \{0\} \text{ and } j \equiv 0 (\mathrm{mod}\, 4)\}$ and such vertices belong to the dominating set $D_0 \subset D$ of the C_{n_0}-layer. For C_{n_1}-layer, let $K = \{k \,|\, k \equiv 2 (\mathrm{mod}\, 4)\}$ and its Cartesian product with the set $I_1 = \{1\}$, that is, $I_1 \times K = \{(i_1, k) | i_1 \in \{1\} \text{ and } k \equiv 2 (\mathrm{mod}\, 4)\}$ and such vertices belong to the dominating set $D_1 \in D$ of the C_{n_1}-layer. For C_{n_2}-layer, let $L = \{l \,|\, l \equiv 1 (\mathrm{mod}\, 4)\}$ and its Cartesian product with the set $I_2 = \{2\}$, that is, $I_2 \times L = \{(i_2, l) \,|\, i_2 \in \{2\} \text{ and } l \equiv 1 (\mathrm{mod}\, 4)\}$ and such vertices belong to the dominating set $D_2 \subset D$ of the C_{n_2}-layer. For C_{n_3}-layer, let $P = \{p \,|\, p \equiv 3 (\mathrm{mod}\, 4)\}$ and its Cartesian product with the set $I_3 = \{3\}$, that is, $I_3 \times P = \{(i_3, p) \,|\, i_3 \in \{3\} \text{ and } p \equiv 3 (\mathrm{mod}\, 4)\}$ and such vertices belong to the dominating set $D_3 \subset D$ of the C_{n_3}-layer. These four sets J, K, L and P will repeat respectively with index i if $i>4$. Hence $D = \bigcup_{i=0}^{m-1} D_i$.

Theorem 2: [S. Klavzar and N. Seifter [9]]: $\gamma(C_4 \square C_n) = n$, where $n \ge 4$.

Theorem 3: For any even integer $m, n \ge 4$ and with $m<n$, $\gamma(C_m \square C_n) = \frac{mn}{4}$ if and only if $2 | \frac{mn}{4}$.

Proof : Let the grid generated by the Cartesian product of the two even cycles C_m and C_n, where $m<n$ and $m \ge 4$, be indexed by i which run in the interval $0 \le i \le m-1$ for the m values. Let the domination set contains the vertices of the form $D = \{(i_0, j), (i_1, k), (i_2, l), (i_3, p), ..., (i_{m-1}, j), ...\}$ where the indices j, k, l and p will repeat respectively for larger m values. Indices are of the type $J = \{j \,|\, j \equiv 0 (\mathrm{mod}\, 4)\}$, $K = \{k \,|\, k \equiv 2 (\mathrm{mod}\, 4)\}$, $L = \{l \,|\, l \equiv 1 (\mathrm{mod}\, 4)\}$ and $P = \{p \,|\, p \equiv 3 (\mathrm{mod}\, 4)\}$ with the intervals $0 \le j \le n-1$, $0 \le k \le n-1$, $0 \le l \le n-1$, and $0 \le p \le n-1$. Working with the four indices, namely; j, k, l and p two cases arise; one when $4|n$ and the other is when 4 does not divide n. In case when 4 divides n, each C_{n_i}-layer contains $\frac{n}{4}$ vertices belonging the domination set $D_i \subset D$. Hence we have total number of vertices $m\left(\frac{n}{4}\right)$,

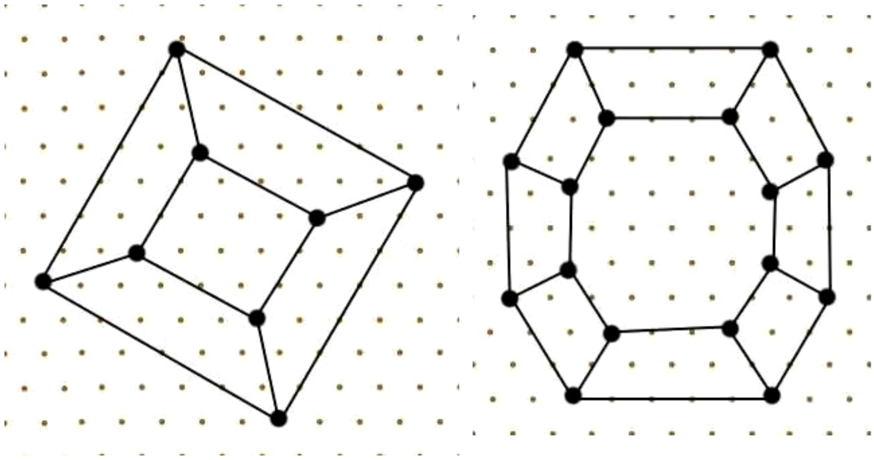

Figure 1. 4-prism and 8-prism graphs .

hence we have $\gamma(C_m \square C_n)=\frac{mn}{4}$ when 4 divides n. In the case, where n is not divisible by 4 then half of the C_{n_i} – layer contains $\frac{m}{2}\left(\left\lceil \frac{n}{4} \right\rceil\right)$ number of vertices belonging the domination set $D_{i=2t-2} \subset D$ and half of the C_{n_i} - layer contains $\frac{m}{2}\left(\left\lfloor \frac{n}{4} \right\rfloor\right)$ number of vertices belonging the domination set $D_{i=2t-1} \subset D$, where t=1,2,... ; consequently we have

$$\frac{m}{2}\left\lceil \frac{n}{4} \right\rceil+\frac{m}{2}\left\lfloor \frac{n}{4} \right\rfloor$$

$$\frac{m}{2}\left(\left\lceil \frac{n}{4} \right\rceil+\left\lfloor \frac{n}{4} \right\rfloor\right)$$

$$\frac{mn}{4}$$

Hence

$$\gamma(C_m \square C_n)=\frac{mn}{4}$$

Prisms graphs are graphs of the type $P_m \square C_n$, where $P_m \square C_n$ is the Cartesian product of the path of length m and the cycle of length n [8]. Let K_2 be the complete graph on two nodes, that is, $K_2=P_2$ then, the Cartesian product $K_2 \square C_n$ is an n-prism, where 4-prism is Cartesian product of $K_2 \square C_4$ which is a cube and the 8-prism is Cartesian product of $K_2 \square C_4$ which is a octagonal prism depicted in Figure 1 above.

Theorem 4. Let $n \geq 4$, and let $4|n$, then $\gamma(C_n \square K_2)=\frac{n}{2}$.

Proof. Let $n \geq 4$, and let $4|n$, then it is proved here that $\gamma(C_n \square K_2)=\frac{n}{2}$. With the basic initial inductive step we will have $\gamma(C_4 \square K_2)=2$. As $4|n$, then $n=4k$ and the k_{th} inductive step would be $\gamma(C_{4k} \square K_2)=2k$ which holds for all $k=1,2,...$. Now leading the last inductive step we have $\gamma(C_{4k+1} \square K_2)=2(k+1)$ which also holds for all values of k. Hence we have $\gamma(C_n \square K_2)=\frac{n}{2}$, $\forall n \geq 4$ with $4|n$.

M. S. Jacobson and L. F. Kinch in [6] proved the limiting value of the domination number $\lim^{m\to\infty}_{n\to\infty} \frac{\gamma(P_m \square P_n)}{mn}=\frac{1}{5}$ as the number m and n gets bigger.

Here, in this note, a construction of a domination set is proposed in lemma 1 above and with this construction following is proposed.

Proposition 5 $\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{\gamma(C_m \square C_n)}{mn} = \frac{1}{4}$.

3. Conclusion

In this note, initial results match with one of the results of S. Klavzar and N. Seifter [9] , stated in theorem 2 above, when $m=4$. The limiting value of the Cartesian product of two cycles C_m and C_n , proved above in theorem 3, is also improved in this note in proposition 5 which was earlier suggested by S. Klavzar and N. Seifter in [9]. A very little work has been done so far on the domination number of the prisms over cycles, C_n , where n is of the form 4k where k= 1, 2. In this note a fresh result is proved in theorem 4 above.

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