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ON CARTESIAN PRODUCT OF CYCLES AND PATHS

*A. D. JUMANI

Department of Mathematics, Shah Abdul Latif University, Khairpur, Pakistan

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Let $G\square H$ denote the Cartesian product of the graphs G and H and let $\gamma(G\square H)$ denote the domination number of the Cartesian product of the two simple graphs G and H. In this note, the domination number of the Cartesian product $C_3\square P_n$, $C_4\square P_n$ and $C_5\square P_n$ is determined; that is $\gamma(C_3\square P_n)=n$, $\gamma(C_4\square P_n)=n$ where $n\geq 1$ and $\gamma(C_5\square P_n)=\left\lfloor \frac{5n-1}{3} \right\rfloor$ where $n\geq 4$.

Keywords : Cartesian product, Domination number, Vizing's conjecture

1. Introduction

Motivation to study graph theory is its ability to represent many situations in life as graph-theoretic models; whether they are natural structure or man-made, like biology, computer science, economics, engineering informatics, linguistics, mathematics, medicine, social science etc. Inspiration to study the domination number of a graph is to make important strategic decisions like placing some service stations in a large network.

A graph G is a triple consisting of a vertex set $V(G)$ that is V, an edge set $E(G)$ that is E and a relation that associates with each edge two vertices called endpoint. A graph is simple if it has no loops and no multiple edges. A set of vertices D of a simple graph G is called dominating *set* if every vertex $w\in V(G)-D$ is adjacent to some vertex $V\in D$. The domination number of a graph G, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of a graph G. A dominating set D with $|D|=\gamma(G)$ is called the minimum dominating set [1]. The Cartesian product $G\square H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and $(a,x)(b,y)\in E(G\times H)$ whenever $x=y$ and $ab\in E(G)$ or $a=b$ and $xy\in E(H)$. For $x\in V(H)$, set $G_x=G\times\{x\}$ and for $a\in V(G)$, set $H_a=\{a\}\times H$, the sets G_x and H_a are called layers of G or H respectively [2].

On domination theory of Cartesian product of graphs; there are two fundamental problems, one is the conjecture of Vizing, which is still open, stated in [2,3] such as $\gamma(G\square H)\geq\gamma(G)\gamma(H)$ that is the domination number of the Cartesian product of the two graphs is at least the product of their domination numbers and for many partial results see [4, 5]. The other problem is to determine the domination number of certain Cartesian products of graphs [4, 7, 8]. Also this problem seems to be a difficult one and even for a subgraph of $D=\{0\times J\}\cup\{2\times L\}$ is NP-complete and the problem itself is also open.

2. Main Results

Lemma 1 : let n be an integer with $n\geq 1$, for $C_3\square P_n$, the domination set $D=\{0\times J\}\cup\{2\times L\}$; where $J=\{j \mid j\equiv 0(\text{mod } 2); j\geq 0\}$ and $L=\{l \mid l\equiv 1(\text{mod } 2); l\geq 0\}$ with $0\leq j\leq n$ and $0\leq l\leq n-1$.

Proof : Let $(0,j)$ and $(2,l)$ be the coordinate pairs of the domination set points Cartesian product of C_3 and P_n, where $n\geq 1$. It is shown here that

* Corresponding author : alidino.jumani@salu.edu.pk

$D = \{0 \times J\} \cup \{2 \times L\}$, where D is the domination set. Let G_x be a layer of the Cartesian product of C_3 and P_n where $x \in H$ and also let H_y be a layer of the Cartesian product of C_3 and P_n, where $y \in G$. Domination set points in layer G_x will take the form of $J = \{j \mid j \equiv 0 (\text{mod } 2); j \geq 0\}$ and $L = \{l \mid l \equiv 1 (\text{mod } 2); l \geq 0\}$. The domination set D is now union of the Cartesian product of the sets $\{0\}$ and $\{J\}$, that is $\{0 \times J\}$; $\{2\}$ and $\{L\}$, that is $\{2 \times L\}$; $D = \{0 \times J\} \cup \{2 \times L\}$. Obvious restriction are imposed on ordinate indices is $j < n$ and $l < n$, which leads to the intervals on indices as $0 \leq j \leq n$ and $0 \leq l \leq n - 1$.

Lemma 2: [7, 9]: $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$

<u>Theorem 3</u> : For any integer $n > 0$, $\gamma(C_3 \square P_n) = n$

Proof: Let D be the dominating set containing the vertices of the form $(0, j)$ and $(2, l)$, where $j \equiv 0 (\text{mod } 2)$ and $l \equiv 1 (\text{mod } 2)$ with $0 \leq j$ and $l \leq n - 1$. From the lemma 1 above the domination set D is an immediate consequence of the construction by Cartesian product of the sets 0 with j and 2 with l respectively and easy to check that $|D| = n$.

Now, in continuation with this, it is proved here that the Cartesian product of C_3 and P_n, such that $\gamma(C_3 \square P_n) \geq n$, whenever $n \geq 1$. Let $x \in V(H)$, the set $G_x = G \square \{x\}$ is a layer of G i.e., G– layer. In Cartesian product of $C_3 \square P_n$ every single G–layer of the product is C_3 and by lemma 2 above $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ and for each G–layer the domination set consists of a single vertex, that is, $\gamma(P_3) = 1$. Hence each G–layer of the product contains only one vertex from the domination set D; consequently n of the G–layers contains n vertices from the domination set and $|D| = n$, which proves that for any integer $n \geq 0$, $\gamma(C_3 \square P_n) = n$.

<u>Lemma 4</u> **:** Let n be an integer with $n \geq 2$, for $C_4 \square P_n$ the domination set $D = \{0 \times J\} \cup \{2 \times L\}$; where $J = \{j \mid j \equiv 0 (\text{mod } 2); j \geq 0\}$, and $L = \{l \mid l \equiv 1 (\text{mod } 2); l \geq 0\}$ with $0 \leq j, l \leq n$.

Proof : Let (0, j) and (2, l) be the coordinate pairs of the domination set of Cartesian product of the cycle C_4 and a path P_n, where $n \geq 2$. It is shown here that the domination set $D = \{0 \times J\} \cup \{2 \times L\}$. Let G_x be a layer of the Cartesian product of cycle C_4 and path P_n, where $x \in H$ and let H_y be also a layer of the Cartesian product of cycle C_4 and path P_n where $y \in G$. Domination set points in the layer H_y will take the form of $j \equiv 0 (\text{mod } 2)$, that is, we say that $J = \{j \mid j \equiv 0 (\text{mod } 2);\ j \geq 0\}$, and $l \equiv 1 (\text{mod } 2)$ that is, we say that $L = \{l \mid l \equiv 1 (\text{mod } 2); l \geq 0\}$. The domination set D is now union of the Cartesian product of the sets 0 and J that is $\{0 \times J\}$ and 2 and L that is $\{2 \times L\}$; that is $D = \{0 \times J\} \cup \{2 \times L\}$. Obvious restrictions are imposed on ordinate indices j and l, which leads to the intervals on indices as $0 \leq j, l \leq n$.

<u>Theorem 5</u> : For any integers $n \geq 2$, $\gamma(C_4 \square P_n) = n$.

<u>Proof</u> : Same line of argument is to be taken as in proving theorem 3. Let D be the dominating set containing the vertices of the form (0, j) and (2, l) where $j \equiv 0 (\text{mod } 2)$, and $l \equiv 1 (\text{mod } 2)$ with $0 \leq j, l \leq n$. The set D is an immediate consequence of the construction by Cartesian product of the sets 0 with J and 2 with L respectively and easy to check that $|D| = n$ again.

Now, in continuation with this, it is proved here that the Cartesian product of C_4 and P_n, such that $\gamma(C_4 \square P_n) \geq n$, whenever $n \geq 2$. Let $x \in V(H)$, the set $Gx = G \square \{x\}$ is a layer of G i.e., G– layer. In Cartesian product of $C_4 \square P_n$ every single G–layer of the product is C_4 and by lemma 2 above $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ and for each G–layer the domination set consists of a single vertex, that is, $\gamma(P_4) = 2$. Hence each G–layer of the product contains only one vertex from the domination set D; consequently n of the G–layers contains n vertices from the domination set and $|D| = n$, which proves that for any integers $n \geq 2$, $\gamma(C_4 \square P_n) = n$.

Lemma 6 : let n be a positive integer with $n \geq 4$ for $C_5 \square P_n$, the domination set
$D = \{0 \times J\} \cup \{2 \times L\} \cup \{4 \times Q\}$; where
$J = \{j \mid j \equiv 0(\text{mod } 2); j \geq 0\}$,
$L = \{l \mid l \equiv 1(\text{mod } 2); l \geq 0\}$, and
$Q = \{q \mid q \equiv 0(\text{mod } 2); q \geq 0\}$.

Proof : Let $(0, j)$, $(2, l)$ and $(4, q)$ be the coordinate pairs of the domination set points Cartesian product of C_5 and P_n where $n \geq 2$. It is shown here that $D = \{0 \times J\} \cup \{2 \times L\} \cup \{4 \times Q\}$, where D is the domination set. Let G_x be a layer of the Cartesian product of cycle C_5 and path P_n where $x \in H$ and let H_y be also a layer of the Cartesian product of C_5 and P_n where $y \in G$. Domination set points in the layer H_y will take the form of $j \equiv 0(\text{mod } 2)$, that is $J = \{j \mid j \equiv 0(\text{mod } 2); j \geq 0\}$, and $l \equiv 1(\text{mod } 2)$, that is $L = \{l \mid l \equiv 1(\text{mod } 2); l \geq 0\}$, and similarly $q \equiv 0(\text{mod } 2)$, that is $Q = \{q \mid q \equiv 1(\text{mod } 2); q \geq 0\}$. The domination set D is now union of the Cartesian product of the sets 0 and J, $\{0 \times J\}$; 2 and L, $\{2 \times L\}$; 4 and Q, $\{4 \times Q\}$; that is $D = \{0 \times J\} \cup \{2 \times L\} \cup \{4 \times Q\}$. Obvious restrictions are imposed on ordinate indices are $j < n$, $l < n$ and $q < n$ which leads to the intervals on indices as $0 \leq j \leq n$, $0 \leq l \leq n$ and also $0 \leq q \leq n$.

Theorem 7 : For any integers $n \geq 4$,

$$\gamma(C_5 \square P_n) = \left\lfloor \frac{5n-1}{3} \right\rfloor$$

Proof : Let D be the domination set containing the vertices of the form $(0, j)$, $(2, l)$ and $(4, q)$, where $j \equiv 0(\text{mod } 2)$, $l \equiv 1(\text{mod } 2)$ and $q \equiv 0(\text{mod } 2)$. By lemma 6 above, D is a construction of the Cartesian product of the sets 0 with J and 2 with L and 4 with Q respectively.

Now, in continuation with this, it is proved here that the Cartesian product of the cycle C_5 and a path P_n, such that $C_5 \square P_n$ the domination set of the product is $\gamma(C_5 \square P_n) = \left\lfloor \frac{5n-1}{3} \right\rfloor$ whenever $n \geq 4$; using the mathematical induction on n and starting with the initial n = 4 we have $\gamma(C_5 \square P_4) = 6$ which holds. Furthering to the k_{th} step we have $\gamma(C_5 \square P_k) = \left\lfloor \frac{5k-1}{3} \right\rfloor$ the relation also holds for the k_{th} step. Then going toward the inductive step $k+1$, and we have $\gamma(C_5 \square P_{k+1}) = \left\lfloor \frac{5k+4}{3} \right\rfloor$ which holds and proves the theorem.

3. Conclusion

Three new results are presented in this note, namely, $\gamma(C_3 \square P_n) = n$, $\gamma(C_4 \square P_n) = n$ and $\gamma(C_5 \square P_k) = \left\lfloor \frac{5k-1}{3} \right\rfloor$. As more general results of the type are NP-complete hence partial results are thee only way have some structure in the domination theory of Cartesian products.

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